2021 ASSESSMENT REPORT
MTM415117 - Mathematics Methods

General Comments

This exam contained a mixture of questions. Some questions were designed to be straightforward, but there were also some difficult discriminating questions. The emphasis was on students understanding the 'big ideas' around the concepts in the course. Therefore, some students found the initial thinking or 'entry point' to the questions challenging.

Similar general comments apply to this year as to previous years. Students need to be confident about using the calculator appropriately in Part Two and equally to be secure in their algebra skills where required throughout the exam. It is important that the precise question being asked is answered for full marks. This was especially the case where descriptive questions required students to refer to prior calculations or algebraic expressions as part of their descriptive answers. These were the threshold marks for a 'C', 'B' or 'A' as determined by the Assessment Panel. All marks are out of 36.

Criterion	Threshold mark for a 'C'	Threshold mark for a 'B'	Threshold mark for an 'A'
Criterion 4: Function Study	16.5	24.5	30.0
Criterion 5: Circular Functions	13.5	23.0	28.5
Criterion 6: Differential Calculus	13.5	23.0	29.0
Criterion 7: Integral Calculus	13.5	22.5	29.0
Criterion 8: Probability	12.0	22.0	28.5

Prospective students are strongly encouraged to read the comments in this assessment report as well as reading the solutions to the exam questions.

Function Study (Part 1)

Students who attempted a question, even when they could not solve the whole problem, gained credit for explanations of their efforts.

Question 1

This question was well done. Most students were able to draw the inverse of the points that made the triangle. Many students were unable to correctly state the required range, finding the domain instead – or the range of a different side. Few students labelled their inverse points, and this may have led to them finding the range of the wrong side.

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Question 2

This question was very well done. Most students recognised the problem style and knew the processes required to complete it. A small number of students did not know the shape of a hyperbola.

Question 3

(a) Most students knew how to set up the binomial expansion. A common problem was not bracketing the terms. i.e. writing $3x^2$ instead of $(3x)^2$ and -2^3 instead of $(-2)^3$. This caused errors in the simplification. A number of students had problems simplifying their expressions.

(b) Many students received part marks for this question. Students could see the transformation required involved a negative and a 3. However, most had problems writing the relationship between $g(x)$ and $f(x)$. this made it difficult for them to interpret the dilation factor. Some students wrote their transformation without giving any justification.

Question 4

(a) The majority of students understood how to set up the required composition of functions. However, few students were able to complete the steps required to simplify the expression to the form given. Most students received part marks.

(b) A number of students gave the required answer, but without any reasoning. A reminder that questions worth 2 or more marks require reasoning/working to be given. Many students did not realise that the domain of $g(f(x))$ = domain of $f(x)$.

(c) Only a small number of students received full marks for this question. Many students did not recognise that the required domain for this sketch was the domain they had found in part (b). Hence, they only needed to find the endpoint and x-intercept. Markers accepted graphs sketched that were consistent with errors made in part (b). A number of students did not recognise the composite function as a cubic. Hence, they attempted to draw some other curve, even though the correct curve was pre-drawn on the axes. Several errors were made attempting to find the x and y intercepts.

Circular Functions (Part 1)

Question 5

(a) Most students correctly determined the period as $\frac{\pi}{3}$ and hence deduced the first asymptote at $\frac{\pi}{6}$. Many didn't interpret the "closest to the origin" correctly and did not include $\frac{-\pi}{6}$ as well.

(b) Only a small proportion of students included a sine sketch with the required domain highlighted. Those that did this generally interpreted the range correctly as $[-\frac{1}{2}, 1]$. Common errors involved $sin\left(\frac{7\pi}{6}\right) = \frac{1}{2}$ or giving $\frac{\sqrt{2}}{2}$ as the upper range value and use of rounded instead of square brackets.

Question 6

Basic angle recognition of $\frac{\pi}{4}$ and that only first and second quadrants needed were both handled well. Different techniques were used by students to consider the adjusted domains and this could be done at the start or toward the end of reasoning to reach the four solutions $-\frac{3}{4}, -\frac{1}{4}, \frac{5}{4}, \frac{7}{4}$. Some struggled handling π in the equation. Two or more than four solutions were often encountered from those unable to work to the specified domain. Any student that attempted to use the general formula had limited success. The depth of understanding associated with this method was inferior to other types of trigonometric reasoning.

Question 7

(a) This question was generally well handled with most students getting the period of 12 months after solstice, midline at 12 hours and amplitude of 5. Some students missed the reflection. Given the grid, the sketches generally had a good cosine shape with the 5 points accurately labelled although the curvature at (0,7) and (12,7) was often not apparent. Some wanted to align numbers to the fourth month of the year instead of solstice references.

(b) Students were expected to state, simplify and solve appropriate equations instead of attempting to approximate from the graph. Most students determined $t = 4$ and $t = 8$ as the extremities of the solutions. The "more than" reference required $(4,8)$ instead of $[4,8]$ or the '<' symbol. If words were used then "between 4 and 8" was deemed correct. Those with a poor understanding of the modelled situation substituted 14.5 for t. Errors from incorrect graphs in part (a) were traced through where appropriate.

Differential Calculus (Part 1)

Question 8

(a) This question was done reasonably well by most students. Marks were deducted by not producing a fully factorised form as required by the question. A common error was writing incorrectly that $(e^{2x})^2$ is e^{4x^2} instead of the correct e^{4x}, or by not cancelling down common factors between the numerator and denominator after the quotient rule was applied.

(b) Many students knew what to do when answering the question. However, they often failed to find y coordinates, merely stating the x values.

Question 9

(a) This question was done very well, however students are reminded that the $\lim\limits_{h \to 0}(your\ expression)$ must be repeated on every line.

(b) Many students made a good attempt at this question but did not relate part (b) to part(a) for full marks as required by the question.

Question 10

This question was well answered. Sometimes extra (invalid) stationary points were included or the shape of the derivative curve was correct except that it was reflected in the x axis. Joining between the stationary points was not as well done.

Question 11

Most students were able to work out $\frac{dA}{d\theta}$ and also that $\frac{dA}{d\theta} = 0$ at a stationary point such as a maximum. Students then needed to state particular gradient values if they chose to justify the maximum through a derivative table. Some common algebraic simplification errors included stating $\cos^2\theta - \sin^2\theta = \frac{1}{2}$ or -1. The question was merely expecting that students would write down the derivative and then justify through substitution or solving an equation that it was a stationary point. Clearly solving the equation was a much more complicated approach. It was also more complicated to use the second derivative method to justify that the stationary point was a maximum – in this case the gradient table method was much easier.

Integral Calculus (Part 1)

Question 12

(a) Part (i) of this question was well done. Errors included not spotting the coefficient of x in the fraction or omitting the '$+c$'. In part (ii), very few students were able to see the link between this part of the question and the one prior to it. Time was often wasted integrating the numerator and denominator separately and then multiplying or dividing the two together. This was awarded no marks.

(b) Students who gained most marks in this question correctly found the log to be the integral, included absolute value signs, and were able to apply a log to index conversation. However, very few students recognised that they needed to justify replacing the absolute value signs with round brackets by stating '$a > 2$' or an equivalent reason.

Question 13

The process that this question required to be solved was generally completed well. It was concerning that students struggled to divide 32 by the negative decimal, while some took the derivative rather than integrating. Most students correctly substituted in the initial temperature coordinate, but many did not apply the index law correctly.

Question 14

(a) This question was well done. Errors included not doubling the integral expression to account for the symmetrical area of the fish below the x axis or putting the terminals in the wrong way around.

(b) Most students had a reasonable attempt at this question, however very few received full marks. Common errors when integrating included incorrectly handling the co-efficient in the cosine function, forgetting to divide by the new power in the square root function, and mistakes with negatives when substituting the terminals in.

Probability (Part 1)

As this was the last section in the paper and it appears that many students were completing the exam questions in order, a significant number of students did not complete all of this section.

Question 15

(a) Many students confused this with a standard 5-trial binomial. Solutions could have been improved, in many cases, by identifying the event with the probability of the event. The probability of 5 tails was often miscalculated as 1/64 rather than 1/32. This error should have been caught while summing the probabilities to check the total is 1.

(b) This question was not answered well. A number of students opted to calculate the probability of a 'head' in less than or equal to four trials (rather than less than four trials). Many students went for the complementary event as an efficient method of solution which was misguided.

Question 16

(a) This question was answered well, given that it was somewhat novel.

(b) Full marks required a precise mean value, but the standard deviation could only be approximately determined from the graph, so allowance was made for a range of plausible answers.

Question 17

Most students were competent in choosing the right formula and substituting correctly, although there were a number who, incorrectly opted for variance = $np(1-p)$. Resolving the answer showed significant issues with decimal arithmetic and squaring of 0.02.

Question 18

(a) Most students correctly calibrated the horizontal axis. Students who correctly calibrated the vertical axis used variance $=np(1-p)$.

(b) Many students described the shape of the curve rather than address the question. Students who successfully answered this question mentioned that p could be any number between 0 and 1 and n was discrete due to being an integer.

Function Study (Part 2)

Most students gained success in this section, recognising the solution style required for the questions. A reminder to students when using the calculator: It is not sufficient to write down an answer and state "from CAS". It is important to show the marker your reasoning and explain your working. The solutions document shows ways answers can be presented.

i.e. Solving the equation $1545 = g(x)$ gives
$x = 1600\ and\ x = 1000$ (this solution is rejected – not in domain)
Thus: Smithies Peak is at $(1600,1545)$
Instead of:
$D = (1600, 1545)$ by CAS

Question 19

(a) This question was very well answered. Almost all students knew how to set up the equation to find the inverse. There were a number of simplification errors – most commonly mishandling the square root sign.

(b) Most students were able to find the domain and range of f(x) correctly. A number of students were unable to apply: domain of $f(x)$ = range of $f^{-1}(x)$ and range of $f(x)$ = domain of $f^{-1}(x)$.

Question 20

(a) Many students found this question challenging. A number of students used log laws to simplify the expression but were unable to put it into the required form. Often not recognising that $\log_2 4 = \log_2(2^2) = 2$.

(b) Many students were able to complete the proof using change of base. A number of students showed understanding of the change of base theorem but did not know how to apply it the problem.

Question 21

This question was well done by most students. There were some calculation errors. Some students did not read the question properly and either, did not label the points of intersection on the graph, or only found the equation of the truncus.

Question 22

(a) Almost all students correctly stated the translations. Common errors included: knowing there was a reflection, but not correctly stating the direction; knowing there was a dilation, but not correctly stating the direction or the factor. A reminder that these are on the Mathematics Methods 4 Information Sheet.

(b) Some students did not attempt this question. Students who did, showed they understood how to solve at least part of the question and achieved some marks. Common errors included: not writing both solutions when solving the equations, and then rejecting the one not in domain. The use of calculator to solve equations without any explanation of what the candidate was finding.

Circular Functions (Part 2)

Question 23

(a) This question was not answered well from both an understanding, and clear labelling, perspective. Some students just pointed towards point B or drew in a horizontal line from point B to the y axis with no indication as to what this meant. Many stated that $\tan\theta = \frac{opp}{adj}$ but didn't make the link to the unit circle where adjacent equals one, giving $\tan\theta = opp$. Labelling slope of the hypotenuse as $\tan\theta$ was deemed as a correct alternative.

(b) Handled well by students with $tan\theta = 2+\sqrt{3}$ or $\theta = tan^{-1}(2+\sqrt{3})$ the common equations stated. Most then successfully used CAS to determine $\theta = \frac{5\pi}{12}$ or $\theta =$ 75 degrees. Part marks awarded for $\tan\theta = \frac{1}{2+\sqrt{3}}$ leading to $\theta = \frac{\pi}{12}$ or $\theta = 15$ degrees or approximate values like $\theta \approx 1.309$.

Question 24

(a) This question was generally well handled, although many students gave little or no reasoning which was required before concluding that $f(x) = -0.5\cos(2x) + 0.5$. Most common error was no negative linked to the reflection. Some ignored the required structure of $f(x) = a\cos(bx) + c$ and examined translations. Some set amplitudes incorrectly as 1.

(b) Again, this question was poorly handled due to a lack of reasoning. The use of CAS, time permitting, provides a check for those students slightly unsure of the extra translation complexity. Many had sign errors linked to algebraic confusion with left or right translations. There are numerous correct responses such as:

$$g(x) = 0.5\sin\left(2\left(x - \frac{\pi}{4}\right)\right) + 0.5$$

$$g(x) = -0.5\sin\left(2\left(x + \frac{\pi}{4}\right)\right) + 0.5$$

$$g(x) = -0.5\sin\left(2\left(x - \frac{3\pi}{4}\right)\right) + 0.5$$

$$g(x) = 0.5\sin\left(2\left(x - \frac{5\pi}{4}\right)\right) + 0.5$$

Some students correctly found a translation of $\frac{\pi}{4}$ units right however incorrectly inserted this into the equation format as shown below:

$g(x) = 0.5\sin\left(2\left(x - \frac{\pi}{2}\right)\right) + 0.5$ which is incorrect compared to $g(x) = 0.5\sin\left(2x - \frac{\pi}{2}\right) + 0.5$

Question 25

(a) This question was handled well by students who drew right angled triangle with hypotenuse from origin to point A. Symmetry leads to $\theta = \frac{\pi}{6}$ and then $\left(cos\left(\frac{\pi}{6}\right), sin\left(\frac{\pi}{6}\right)\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Part marks given to those setting $\theta = \frac{\pi}{3}$ and correctly linking to unit circle calculations.

(b) This question was not answered well with many students giving no response at all. Those that recognised need for additional right-angled triangle drawn to point H and use of $\frac{1}{2}$ from part a) for opposite side could show that $\frac{\sqrt{3}}{3}$ as inner radius. Many observed that $tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$ which is not a sufficient approach when asked to "show".

Question 26

This question was handled extremely well. Most students recognised that "no vertical translation" meant that C=$\left(\frac{\pi}{2}, 0\right)$. The period at 3π then leads to $y = tan\left(\frac{1}{3}\left(x - \frac{\pi}{2}\right)\right)$. Most students then went onto successfully calculate values for the points A and B. Some stopped at $y = tan\left(\frac{1}{3}\left(x - \frac{\pi}{2}\right)\right)$ which is the usual endpoint for these sorts of problems. The most common error was associated with the $y = tan\left(3\left(x - \frac{\pi}{2}\right)\right)$ conclusion which led to an undefined calculation. Some penalty was associated with decimal approximations instead of $\pm\frac{\sqrt{3}}{3}$ for A and B respectively.

Differential Calculus (Part 2)

Question 27

(a) This question was done very well.

(b) This question was done reasonably well. However, some students averaged the derivative values where $x = 1$ and where $x = 3$ which coincidentally led to the correct answer but were awarded no marks as the approach was entirely incorrect.

Question 28

Students are encouraged to check their derivative expression with the calculator to identify quickly if they have made algebraic mistakes. There were many opportunities to award part marks for errors carried forward. It was important the students rejected $x = -3$ after solving the equation as it is invalid, in order to gain full marks.

Question 29

This question confused many students with the principal area of confusion relating to the idea of what 'magnitude' means. To gain the full 4 marks, stating correct domain with appropriate reasoning/justification was required.

Question 30

This question was answered very well, with substantial marks for errors carried forward awarded to many students who did not gain full marks. A common error was stating the full algebraic expression for the derivative rather than substituting in the appropriate x value so that the gradient becomes a number.

Question 31

(a) Many part marks were awarded to this question as students in the main were able to show that $f'(d)$ and $g'(d)$ were the same value, thus proving smoothness. However, many students also failed to realise that they needed to prove continuity as well, i.e. that $f(x) = g(x)$ at $x = d$.

(b) Students mainly did well in this part, but some of them confused the idea of a 'rate' with a 'function value'.

Integral Calculus (Part 2)

In this section, there was considerable evidence of students' poor arithmetic skills. If examiners were deducting whole marks for this, students would have difficulty gaining marks. Where possible, students are advised to use their calculator for arithmetic because often mistakes were made in the calculator when they did not utilise the technology available to them.

Question 32

(a) Most students recognised that it was necessary to split the expression into two sections and make the area under the x axis positive. Mistakes included incorrect use of absolute value signs and failing to write dx.

(b) This question proved more challenging than expected. Most students correctly identifying that the area under the x axis is larger than the area above it, however writing 'negative area' is entirely incorrect. Full marks were awarded when the explanation recognised that the difference between these areas is precisely one unit of area.

Question 33

This question was generally not answered well. Students were able to integrate the constant correctly to obtain a value of 6 but weren't sure where to go from there. Some students have not encountered many questions where the terminals of an integral have been switched and were unable to figure this out in this exam situation.

Question 34

Students who integrated the velocity between 0 and 4 had great success with this question. Those who found a formula for distance often assumed that the constant was zero without proving this by substituting the starting co-ordinate (0,0). Another common error was to use the value 32 that was provided, rather than calculating it for themselves – not an advisable approach to 'show that' questions.

Question 35

(a) This question was well done. Common errors included answering to too many decimal places and omitting units in the final answer, as well as subtracting $f(x)$ from $g(x)$ without using brackets. Calculating this resulted in an incorrect negative answer.

(b) Most were able to follow through with this part of the question, with some occasional errors using division rather than multiplication or omitting cubic metres.

Question 36

This was a challenging question, and many students did not attempt it. Those who did were generally successful. They rearranged the linear equation and identified the gradient, equating this to the gradient function of the hyperbolas. Others found the integral and equated this to the linear line, which proved difficult to solve.

Probability (Part 2)

Given students had the opportunity in Part Two to use their calculators, it was notable that some students chose not to on occasions, often to their detriment.

Question 37

This was an accessible entry to the section for most students. Many gained full marks. Some students gained part marks for finding $E(X)$ and stopping. A number of students lost marks by not providing evidence for the value for $E(X^2)$, neglecting to either square the expected value or to take the square root to find the standard deviation.

Question 38

This question was answered well. Most students correctly identified the variables and set up the equation for the margin of error correctly. Most students who lost part marks did so by not rounding (or rounding down) the answer to reflect the correct whole number of tennis balls.

Question 39

(a) This was reasonably well understood by students, but a significant number of students missed full marks by not starting their efforts with the appropriate statement or diagram describing the distribution. Typically, their answer commenced with the relevant calculator instruction and proceeded to the correct answer from there.

(b) Students found this not as accessible as part (a) and fewer gained full marks, Again, there was often an absence of a diagram or explanation of the distribution and too little evidence of process (considering that this is a calculator section). Some students used 'trial and error' as their method of choice with successful results, others used their calculators to solve for b via for example `solve(normcdf(110,b,15,100)=0.15)`. The other successful approach was to infer $\Pr(\text{IQ} > b) = 0.10249$ and then use the inverse normal.

Question 40

(a) Most students received part or full marks from this question. Given that the answer was provided, marks were deducted if there was no statement that they solved for p or other evidence to show process between their initial statement and their conclusion that $p = 0.23$.

(b) Part (i) was successfully completed by most students; however, if the general accepted standard is 4 decimal places, there was a significant amount of 'excessive rounding' going on which, of course, flowed through to the subsequent question. The students who calculated the individual probabilities for 0, 1 and 2 diners and subtracted from one weren't as successful as those who used the distribution function. The increased opportunity for arithmetic errors appeared to take its toll. In part (ii), not as many students gained full marks in this part. Again, not making a statement describing the new distribution was common. Other students using the original p value or found the probability for 'greater than" five nights.

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External Assessment 2021

Attach your candidate label here

SOLUTIONS

MATHEMATICS METHODS

MTM415117

Part 1

Pages	24
Questions	18
Information Sheet	1

Reading time: 15 minutes – you may begin writing during this time

Suggested working time: 80 minutes

Instructions

Calculators are not allowed to be used.

Part 1 will be collected after 80 minutes.

- There are **five (5)** sections to this exam paper.
- Answer **all** questions and **all** parts within each question.
- Write your answers in the spaces provided in this exam paper.
  - Spare diagrams have been provided at the end of each section. Indicate in the box provided if you have used the spare diagram.
- During the first 80 minutes you may move onto Part 2, but you **cannot** use your calculator until told by your Supervisor(s).
- All answers must be written in **English**.
- You **must** make sure your answers address:
  - Criterion 4 understand polynomial, hyperbolic, exponential and logarithmic functions.
  - Criterion 5 understand circular functions.
  - Criterion 6 use differential calculus in the study of functions.
  - Criterion 7 use integral calculus in the study of functions.
  - Criterion 8 understand binomial and normal probability distributions and statistical inference.

Marker use	
	16
	16
	16
	16
	16

Additional Exam Instructions

For questions worth **one (1)** mark, you are not required to show workings. Markers will look at the presentation of the answer(s) and at the argument(s) leading to the final answer(s).

For questions worth **two (2)** or more marks **you are required** to show relevant workings.

Marks will be allocated:

- according to the degree to which workings convey a logical line of reasoning
- for suitable justifications and explanations of methods and processes when requested.

Guide to Exam Structure

	Sections	Questions available	How many questions to answer	Suggested working time	Marks available
Part 1	Section A	4	4	16 minutes	16
	Section B	3	3	16 minutes	16
	Section C	4	4	16 minutes	16
	Section D	3	3	16 minutes	16
	Section E	4	4	16 minutes	16
Total		**18**	**18**	**80 minutes**	**80**
Part 2	Section A	4	4	20 minutes	20
	Section B	4	4	20 minutes	20
	Section C	5	5	20 minutes	20
	Section D	5	5	20 minutes	20
	Section E	4	4	20 minutes	20
Total		22	22	100 minutes	100
Total		**40**	**40**	**180 minutes (3 hours)**	**180**

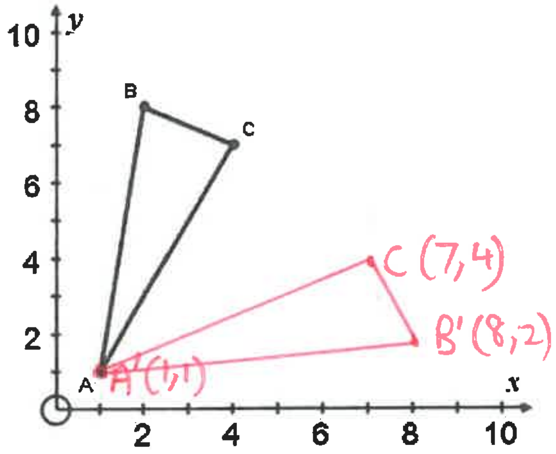
Section A

- Answer **all** questions in this section.
- This section assesses **Criterion 4.**

Question 1

Marker use

Three linear functions are sketched below over restricted domains to form the triangle ABC.



Spare diagram used (✓)

On the same grid, draw the triangle $A'B'C'$ that results from the **inverse** of each point.

State the range of the side $\overline{A'B'}$.

Range of y coordinate for $\overline{A'B'}$ is $y \in [1,2]$

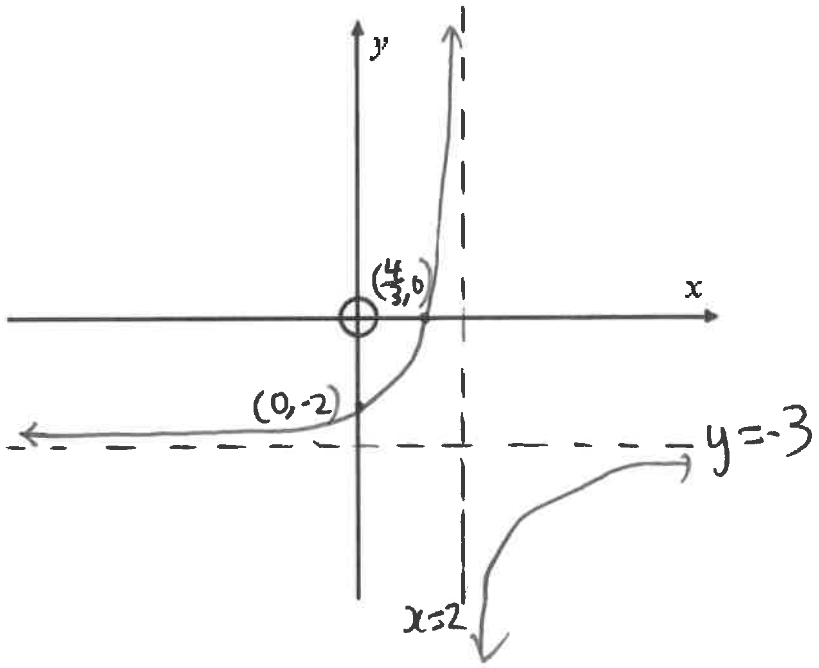
/2

Section A continues

Question 2

Sketch the graph of $y = \frac{-2}{(x-2)} - 3$ on the axes provided.

Label all axes intercepts and asymptotes.



Spare diagram used (✓)

Marker use

/3

V.a. at $x = 2$

h.a. at $y = -3$

y-intercept when $x = 0 \Rightarrow y = \frac{-2}{-2} - 3 = -2$

x-intercept when $y = 0 \Rightarrow 0 = \frac{-2}{x-2} - 3$

$3 = \frac{-2}{x-2}$

$x - 2 = \frac{-2}{3}$

$x = 2 - \frac{2}{3} = \frac{4}{3}$

Marker use

Question 3

a) Determine the binomial expansion of $f(x) = (3x-2)^3$.
Fully simplify your solution.

1
1 1
1 2 1
1 3 3 1

/2

$$f(x) = 1(3x)^3(-2)^0 + 3(3x)^2(-2)^1 + 3(3x)^1(-2)^2 + 1(3x)^0(-2)^3$$
$$= 27x^3 - 3(9)(2)x^2 + 3(3)(4)x - 8$$
$$= 27x^3 - 54x^2 + 36x - 8$$

b) Hence, determine the graphical transformations from $f(x)$ to obtain $g(x)$ where:
$g(x) = -9x^3 + 18x^2 - 12x + \frac{8}{3}$.

/3

$g(x) = -\frac{1}{3}f(x) = -\frac{1}{3}(27x^3 - 54x^2 + 36x - 8)$ by inspecting coefficients

$\therefore$ Graphical transformations from $f(x) \rightarrow g(x)$ are:

- reflection in the x axis
- dilation by factor $\frac{1}{3}$ ~~parallel to the y axis~~ in the y direction / along the y axis / away from the x axis

Section A continued

Question 4

If $f(x)=3\ln(x-2)+5$ and $g(x)=e^{(x-3)}-1$ then the composite function $g\left[f(x)\right]$ exists for a restricted domain.

a) Show that $g\left[f(x)\right]=e^2(x-2)^3-1$.

$g(f(x)) = e^{(f(x)-3)}-1$

$= e^{(3\ln(x-2)+5-3)}-1$ for $x>2$

$= e^{3\ln(x-2)+2}-1$

$= e^2 e^{\ln((x-2)^3)}-1$

$= e^2(x-2)^3-1$ Since $e^{\ln(A)}=A$

($e^{\square}$ & $\ln(\square)$ are inverses of each other)

b) Determine the maximal domain of $g\left[f(x)\right]$.

$g(f(x))$ is defined for when $f(x)$ is defined

Since g has a domain and range of $\mathbb{R}$.

$f(x)$ defined for $x\in(2,\infty)$ $\therefore$ maximal domain for $g(f(x))$ is $x\in(2,\infty)$

c) $y=e^2(x-2)^3-1$ is graphed below.

On the same axes sketch the composite function $g\left[f(x)\right]$.

Label any **relevant** intercepts and endpoints.

When $x>2$,

$g(f(x))>e^2(0)^3-1$

$g(f(x))>-1$

so endpoint (not included) is $(2,-1)$

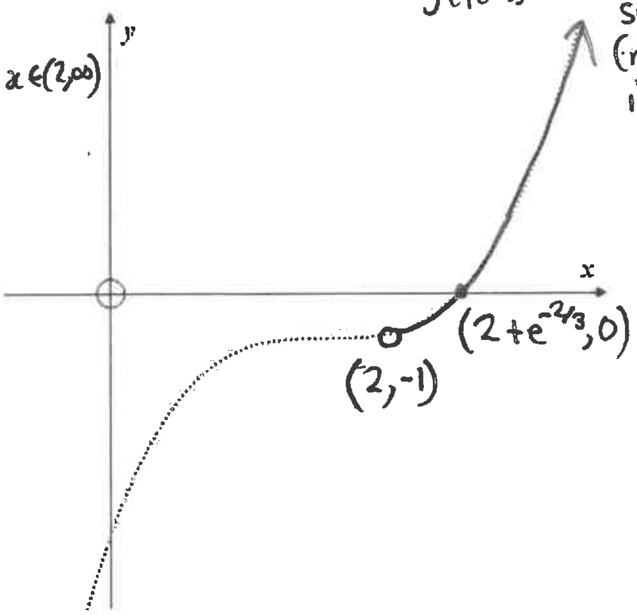
- No relevant y-intercept considering the maximal domain $x\in(2,\infty)$
- x-intercept when $y=0$

$0=e^2(x-2)^3-1$

$\frac{1}{e^2}=(x-2)^3$

$e^{-2/3}=x-2$

$2+e^{-2/3}=x$



Spare diagram used (✓)

Marker use

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/2

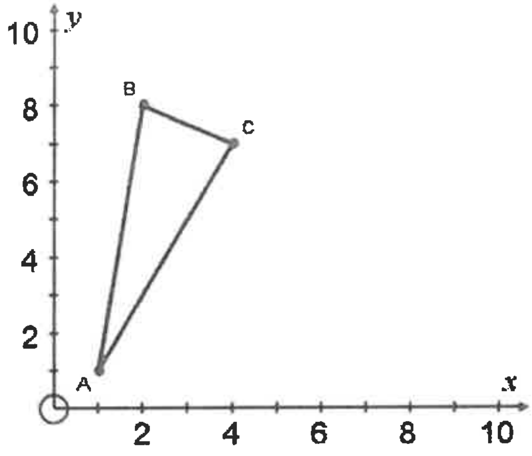
/2

Total C4

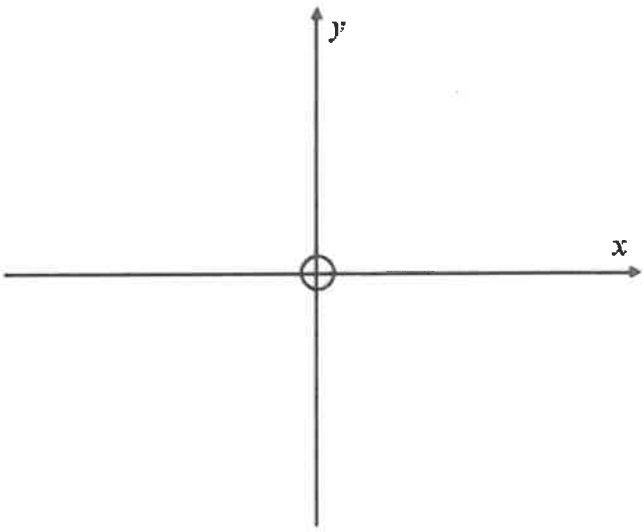
/16

Spare Diagrams

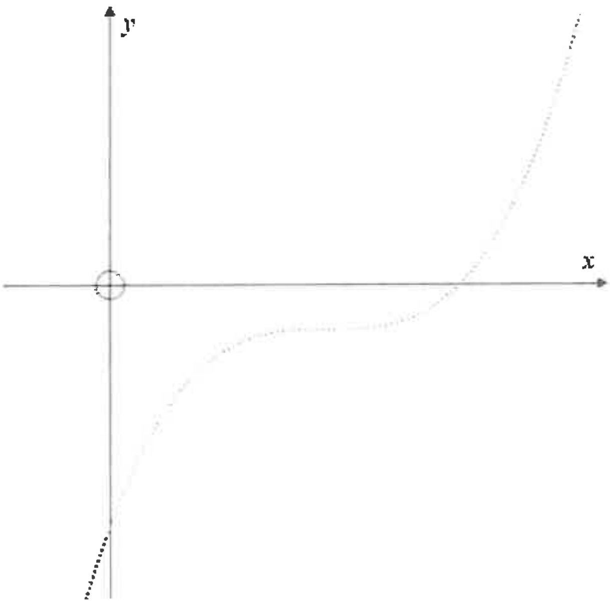
Question 1



Question 2



Question 4c



Section B

- Answer **all** questions in this section.
- This section assesses **Criterion 5**.

Question 5

Marker use

a) Find the vertical asymptotes closest to the origin when $f(x)=\tan(3x)$.

/2

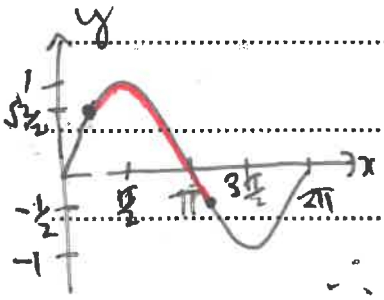
$\tan(3x)$ is $\tan(x)$ dilated by factor $\frac{1}{3}$ in x direction, i.e. period $=\frac{\pi}{3}$

Asymptotes for $\tan(x)$ are $x=\frac{-\pi}{2}$ and $x=\frac{\pi}{2}$ closest to origin

$\therefore$ Asymptotes for $\tan(3x)$ are $x=\frac{-\pi}{6}$ and $x=\frac{\pi}{6}$ closest to the origin

b) Determine the range of $y=\sin(x)$, $x\in\left[\frac{\pi}{4},\frac{7\pi}{6}\right]$.

/3



$\sin\left(\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$ (1st quadrant)

$\sin\left(\frac{7\pi}{6}\right)=\sin\left(\pi+\frac{\pi}{6}\right)=\frac{-1}{2}$ (2nd quadrant)

$\therefore$ Range is $y\in\left[\frac{-1}{2},1\right]$

Section B continues

Marker use

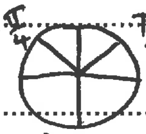
Question 6

Solve $\sin(\pi x+\pi)=\frac{\sqrt{2}}{2}$ for $x \in [-1,2]$.

/ 4

$\sin(\pi(x+1)) = \frac{\sqrt{2}}{2}$

Basic angle is $\frac{\pi}{4}$



$\therefore \pi(x+1) = \frac{\pi}{4} + 2k\pi, \frac{3\pi}{4} + 2k\pi$ where $k \in \mathbb{Z}$

$\pi(x+1) = \frac{\pi+8k\pi}{4}, \frac{3\pi+8k\pi}{4}$

$x+1 = \frac{1+8k}{4}, \frac{3+8k}{4}$ for $-\frac{4}{4} < x < \frac{8}{4}$

$x = \frac{-3+8k}{4}, \frac{-1+8k}{4}$

$x = \underbrace{-\frac{3}{4}, -\frac{1}{4}}_{k=0}, \underbrace{\frac{5}{4}, \frac{7}{4}}_{k=1}$

or another method:

Considering domain: $-1 \le x \le 2$

$0 \le x+1 \le 3$

$0 \le \pi(x+1) \le 3\pi$

Using basic angle $\frac{\pi}{4}$, solutions are:

$\pi(x+1) = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{4}+2\pi, \frac{3\pi}{4}+2\pi$

$\pi(x+1) = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$

$x+1 = \frac{1}{4}, \frac{3}{4}, \frac{9}{4}, \frac{11}{4}$

$x = -\frac{3}{4}, -\frac{1}{4}, \frac{5}{4}, \frac{7}{4}$

Section B continued

Question 7

Marker use

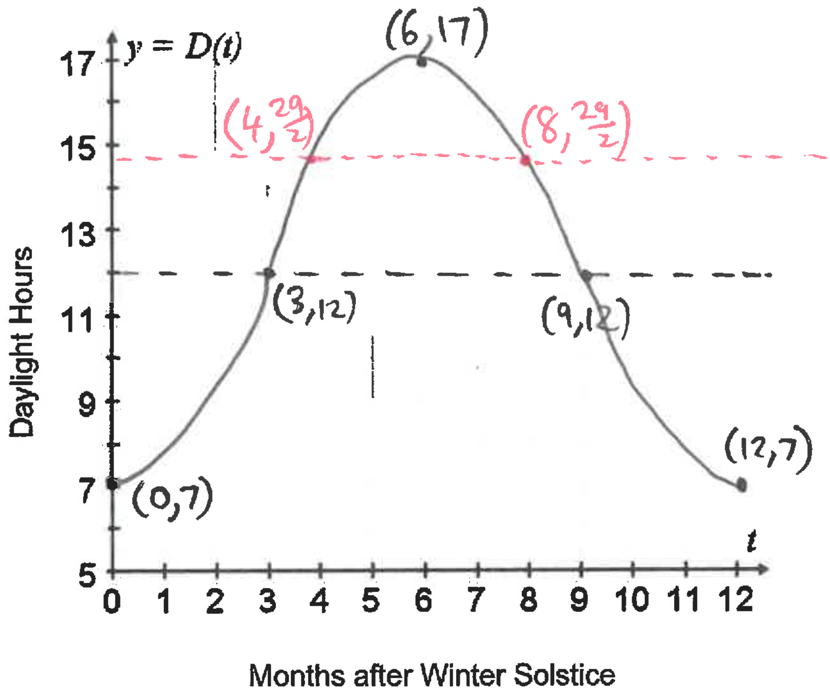
Daylight hours for Macquarie Island can be modelled by the function $D = -5\cos\left(\frac{\pi t}{6}\right) + 12$,

where t represents the months after the winter solstice which occurs at $t = 0$.

a) Complete a sketch for one period on the axes provided below.

Label the coordinates when $t = 0, 3, 6, 9$ and 12.

/4



Spare diagram used (✓) ☐

Midline = 12, Amplitude = 5, graph shape (reflection in x axis)

Period = $\frac{2\pi}{\frac{\pi}{6}} = 12$

b) Determine when Macquarie Island has more than $14\frac{1}{2}$ hours of daylight. $= \frac{29}{2}$ hours of daylight

/3

Need to first solve $-5\cos\left(\frac{\pi t}{6}\right) + 12 = \frac{29}{2}$

$-5\cos\left(\frac{\pi t}{6}\right) = \frac{5}{2}$

$\cos\left(\frac{\pi t}{6}\right) = \frac{-1}{2}$

$\therefore \frac{\pi t}{6} = \frac{2\pi}{3}, \frac{4\pi}{3}$ $\quad \pi - \frac{\pi}{3}$, $\pi + \frac{\pi}{3}$

$t = \frac{2\pi}{3} \times \frac{6}{\pi}, \frac{4\pi}{3} \times \frac{6}{\pi}$

$t = 4, 8$

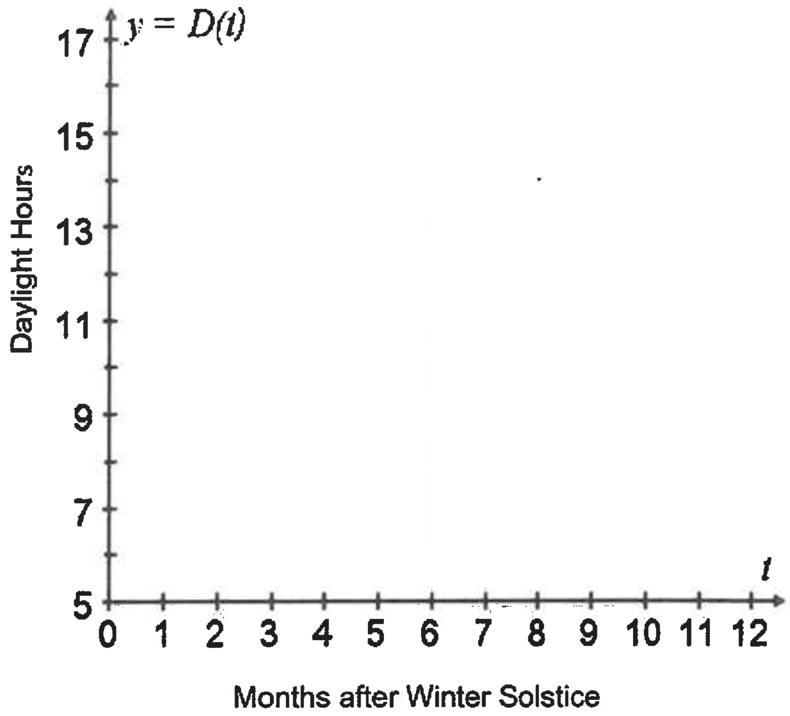
$\therefore$ More than $14\frac{1}{2}$ hours of daylight for $4 < t < 8$
i.e. between the 4th & 8th month after the winter solstice.

Total C5

/16

Spare Diagram

Question 7



Section C

- Answer **all** questions in this section.
- This section assesses **Criterion 6**.

Question 8

Marker use

a) Given $f(x)=\dfrac{x^2}{e^{2x}}$, use the quotient rule to determine a simplified factorised expression for $f'(x)$.

/3

$$f'(x) = \frac{e^{2x}.2x - x^2.2e^{2x}}{(e^{2x})^2}$$

$$= \frac{e^{2x}(2x-2x^2)}{e^{4x}}$$

$$= \frac{2x(1-x)}{e^{2x}}$$

b) Hence, determine the stationary point(s) for $f(x)$.

No need to justify the nature of stationary points.

/2

Stat. points when $f'(x)=0$

$$\frac{2x(1-x)}{e^{2x}}=0$$

$x=0, 1$ since denominator $\neq 0$ always

$\therefore$ Points are $\left(0, \frac{0^2}{e^{2(0)}}\right) = (0,0)$

and $\left(1, \frac{1^2}{e^{2(1)}}\right) = \left(1, \frac{1}{e^2}\right)$ (or $(1, e^{-2})$)

Section C continues

Section C continued

Marker use

Question 9

a) Determine $\lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$

$= \lim_{h\to 0}\left\{ \frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h} \right\}$

$= \lim_{h\to 0}\left\{ \frac{\cancel{h}(2x+h)}{\cancel{h}} \right\}$

$= \lim_{h\to 0}\{2x+h\}$

$= 2x$

2

b) Explain what the limit from part a) represents.

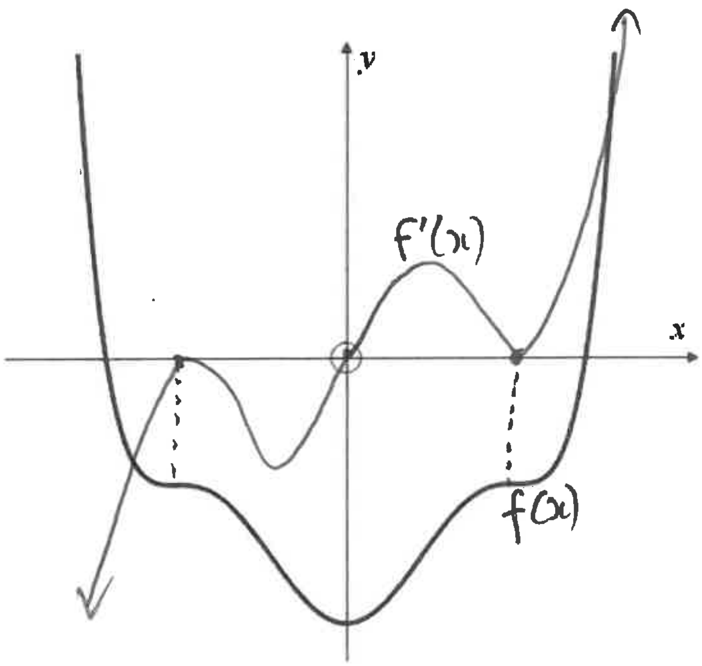
This represents the gradient function $f'(x)$ for the function $f(x)=x^2$ (i.e. the gradient of the tangent at a point on the curve $y=x^2$). Part (a) is the first principles method of determining derivative, which is the instantaneous rate of change.

1

Question 10

The function below is symmetrical about the y axis, has two stationary points of inflection and a local minimum. Sketch a possible derivative function on the same axes.

3



Spare diagram used (✓) ☐

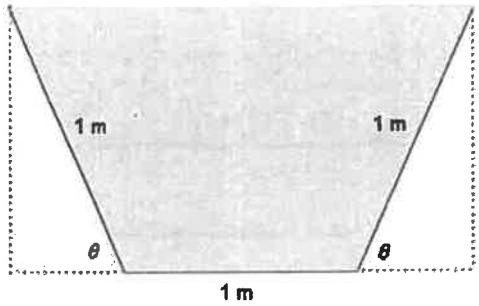
Section C continues

Section C continued

Question 11

A metal wire three metres long is bent up at angles of θ so that 1 metre sections form the base and sides of a trapezoid.

The profile formed is shown below.



Note: $0 \leq \theta \leq \frac{\pi}{2}$ for this situation

The **shaded** trapezoidal area is given by $A = \sin(\theta) + \cos(\theta)\sin(\theta)$.

Show, with justification, that $\theta = \frac{\pi}{3}$ maximises the area A.

/5

$$\frac{dA}{d\theta} = \cos\theta + \cos\theta \cdot \frac{d\sin\theta}{d\theta} + \frac{d\cos\theta}{d\theta} \cdot \sin\theta$$ by product rule

$$= \cos\theta + \cos^2\theta - \sin^2\theta$$

Min/max occurs when $\frac{dA}{d\theta} = 0$

Solving equation

$$\cos\theta + \cos^2\theta - \sin^2\theta = 0$$
$$\cos\theta + \cos^2\theta - (1 - \cos^2\theta) = 0$$
$$\cos\theta + 2\cos^2\theta - 1 = 0$$
$$2\cos^2\theta + \cos\theta - 1 = 0$$
$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$

$\therefore \cos\theta = \frac{1}{2}$ or $\cos\theta = -1$

$\therefore \theta = \pi$ rejected (outside domain)

$\theta = \frac{\pi}{3}$ for required domain

Substituting in $\theta = \frac{\pi}{3}$

When $\theta = \frac{\pi}{3}$,

$$\frac{dA}{d\theta} = \cos\left(\frac{\pi}{3}\right) + \cos^2\left(\frac{\pi}{3}\right) - \sin^2\left(\frac{\pi}{3}\right)$$
$$= \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2$$
$$= \frac{1}{2} + \frac{1}{4} - \frac{3}{4} = 0$$ as required

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\frac{dA}{d\theta}$	2	0	-1
	/	—	\

$1 + 1^2 - 0^2$ $\qquad$ $0 + 0^2 - 1^2$

$\therefore$ Local maximum at $\theta = \frac{\pi}{3}$

Other possible values for gradient table from Information Sheet

$\theta = \frac{\pi}{4}, \frac{dA}{d\theta} = \frac{\sqrt{2}}{2}$

$\theta = \frac{\pi}{6}, \frac{dA}{d\theta} = \frac{\sqrt{3}+1}{2}$

Alternative method of justification

$$\frac{d^2A}{d\theta^2} = -\sin\theta + 2(\cos\theta)(-\sin\theta) - 2(\sin\theta)(\cos\theta)$$
$$= -\sin\theta - 4\sin\theta\cos\theta$$

At $\theta = \frac{\pi}{3}$, $\frac{d^2A}{d\theta^2} = -\frac{\sqrt{3}}{2} - 4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)$

$$= -\frac{3\sqrt{3}}{2} < 0$$

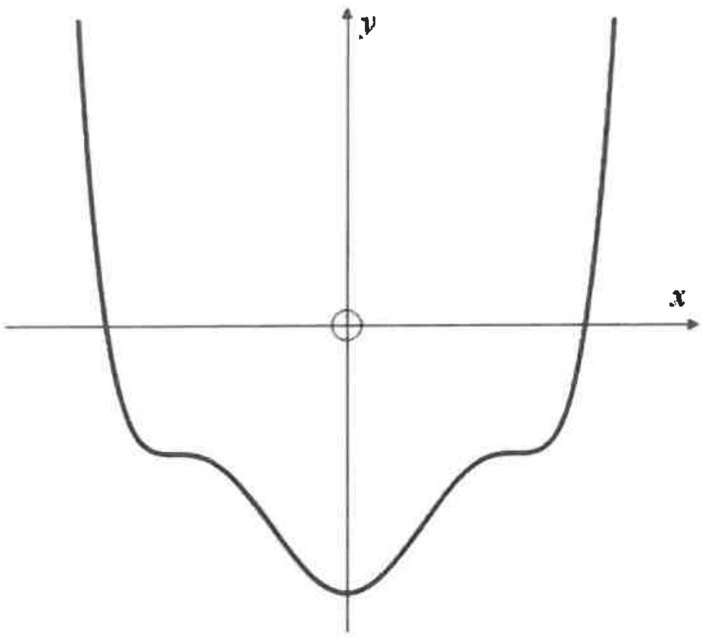
$\therefore$ Local maximum at $\theta = \frac{\pi}{3}$

Total C6

/16

Spare Diagram

Question 10



Section D

- Answer **all** questions in this section.
- This section assesses **Criterion 7**.

Question 12 | Marker use

a) i. Determine $\int (2x-1)^4 \mathrm{d}x$.

$= \frac{(2x-1)^5}{2(5)} + c$ where c is a constant

$= \frac{(2x-1)^5}{10} + c$

/2

ii. Hence, determine $\int \frac{(2x-1)^5}{(1-2x)} \mathrm{d}x$.

$= \int \frac{-(2x-1)^5}{2x-1} dx$

$= -\int (2x-1)^4 dx \quad = -\frac{(2x-1)^5}{10} + c_2$ where c_2 is a constant

/1

b) Given $\int_2^a \frac{1}{(x-1)} \mathrm{d}x = 1$, determine an exact value for a, where a > 2.

$\left[\ln|x-1|\right]_2^a = 1$

$\ln(a-1) - \ln(2-1) = 1$ since $a>2$

$\ln(a-1) = 1$

$a-1 = e$

$a = e+1$

/3

Section D continues

Marker use

Question 13

The temperature of a soup varies at a rate modelled by $\frac{dT}{dt} = 32e^{-0.4t}$ where T is the temperature (^{0}C) and t is the time in minutes after the temperature begins to change.

Determine an equation for T given at $t = 0$ the temperature was 20^0C.

/4

$T = \int \frac{dT}{dt}\, dt$

$= \int 32e^{-0.4t}\, dt$

$= 32\left(\frac{e^{-0.4t}}{-0.4}\right) + c$ when c is a constant

$= \frac{-32\times5}{2} e^{-0.4t} + c$

$T = -80e^{-0.4t} + c$

When $t=0$, $T=20$ $\therefore$ $20 = -80e^{-0.4(0)} + c$

$20 = -80(1) + c$

$\therefore c = 100$

$\therefore$ Equation is $T = -80e^{-0.4t} + 100$

Section D continued

Marker use

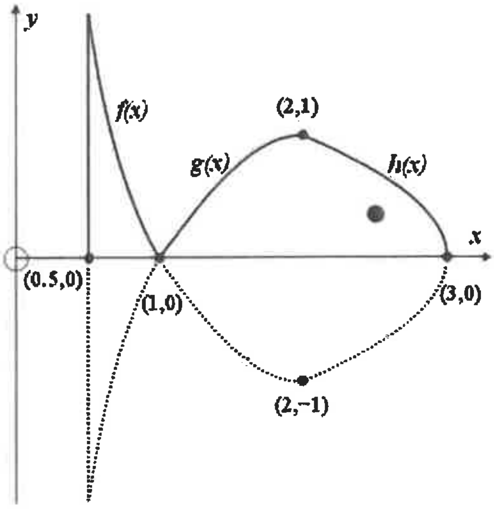
Question 14

The "fish" sketch below has an enclosed area between $x = 0.5$, $f(x)$, $g(x)$, $h(x)$ and **reflections** of these functions about the x axis where:

$f(x) = \dfrac{2}{x} - 2 \qquad x \in [0.5, 1]$

$g(x) = -\cos\left(\dfrac{\pi x}{2}\right) \qquad x \in [1, 2]$

$h(x) = \sqrt{3 - x} \qquad x \in [2, 3]$



a) State a definite integral expression for the total area of the enclosed shape.

Total area $= 2\left(\int_{0.5}^{1} f(x)\,dx + \int_{1}^{2} g(x)\,dx + \int_{2}^{3} h(x)\,dx\right)$

using reflection symmetry

/2

b) Determine an exact value for this area.

Area $= 2\left(\int_{0.5}^{1}\left(\frac{2}{x}-2\right)dx + \int_{1}^{2}\left(-\cos\left(\frac{\pi x}{2}\right)\right)dx + \int_{2}^{3}(3-x)^{\frac{1}{2}}dx\right)$

$= 2\left(\left[2\ln|x| - 2x\right]_{0.5}^{1} + \left[-\frac{\sin\left(\frac{\pi x}{2}\right)}{\pi/2}\right]_{1}^{2} + \left[\frac{(3-x)^{3/2}}{-3/2}\right]_{2}^{3}\right)$

$= 2\left(\left[2\ln|x| - 2x\right]_{0.5}^{1} + \left[-\frac{2\sin\left(\frac{\pi x}{2}\right)}{\pi}\right]_{1}^{2} + \left[\frac{-2(3-x)^{3/2}}{3}\right]_{2}^{3}\right)$

$= 2\left(2(0) - 2(1) - 2\ln(0.5) + 2(0.5) - \frac{2\sin(\pi)}{\pi} + \frac{2\sin\left(\frac{\pi}{2}\right)}{\pi} - \frac{2(0)}{3} + \frac{2(1)}{3}\right)$

$= 2\left(-2 + 2\ln(2) + 1 - 0 + \frac{2}{\pi} + \frac{2}{3}\right)$

$= 2\left(-1 + 2\ln(2) + \frac{2}{\pi} + \frac{2}{3}\right)$

$= -2 + 4\ln(2) + \frac{4}{\pi} + \frac{4}{3}$

$= \left(4\ln(2) + \frac{4}{\pi} - \frac{2}{3}\right)$ units^2

/4

Total C7

/16

Section E

- Answer **all** questions in this section.
- This section assesses **Criterion 8**.

Question 15

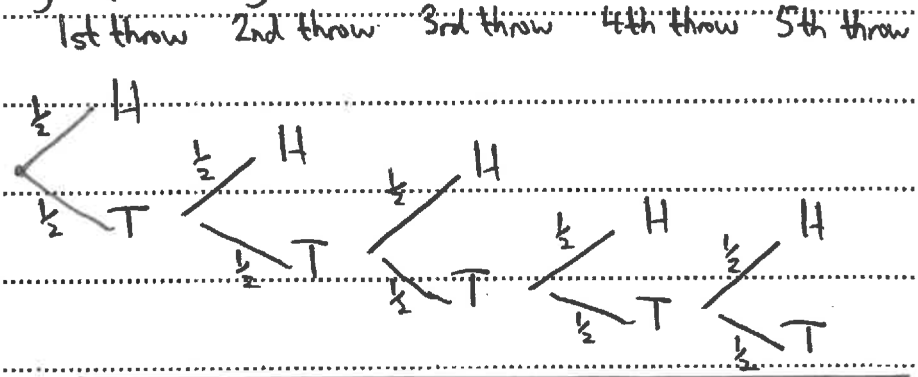
Marker use

Gracie tosses a coin until she either gets a head or completes five throws.

a) Determine the probability of the six possible events that make up the sample space. Show that the sum of these probabilities equals one. /3

Use H and T to denote heads and tails respectively.

A way of thinking through this is with a tree diagram (many different approaches are possible).



x	H	TH	TTH	TTTH	TTTTH	TTTTT
$Pr(X=x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{32}$

x = outcome as per tree diagram

$$\therefore \sum Pr(X=x) = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{32}$$

$$= \frac{16+8+4+2+1+1}{32} = \frac{32}{32} = 1 \text{ as required.}$$

b) Determine the probability of getting a "head" in less than four throws. /1

We are after $Pr(H) + Pr(TH) + Pr(TTH)$

$$= \frac{1}{2}+\frac{1}{4}+\frac{1}{8}$$

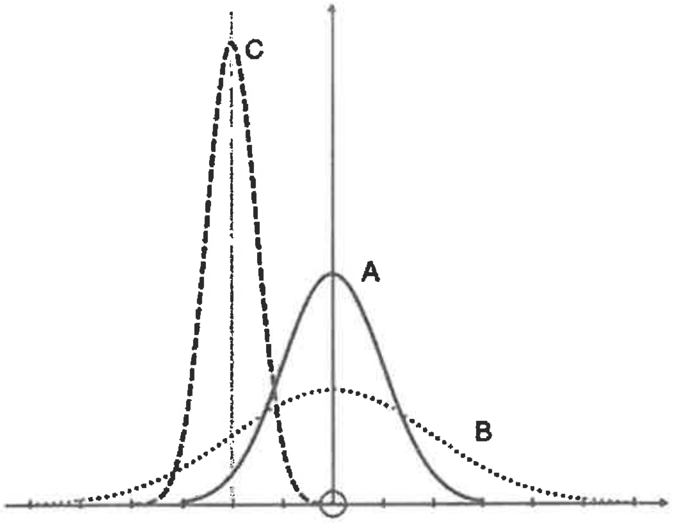
$$= \frac{4+2+1}{8} = \frac{7}{8}$$

Section E continues

Marker use

Question 16

Three normal distributions, labelled A, B and C are shown in the diagram below.



Spare diagram used (✓)

a) **Label** the horizontal scale with appropriate integer values, given A represents the standard normal distribution where the mean is zero and standard deviation is one.

1

b) Hence, state the means and standard deviations for the distributions B and C.

For distribution B: $\mu = 0$ (same as A), $\sigma \simeq 2$ (Somewhere between 1.5 and 2 inclusive)

For distribution C: $\mu = -2$, $\sigma \simeq \frac{1}{2}$ (Somewhere between $\frac{1}{3}$ and $\frac{2}{3}$ inclusive).

2

Question 17

A distribution for $\hat{p}$ has a standard deviation of 0.02 and a mean of 0.80

Determine the sample size.

3

$$E(\hat{p}) = p = 0.80$$

$$\sigma_{\hat{p}} = \sqrt{\frac{p(1-p)}{n}} = 0.02$$

$$\therefore 0.02 = \sqrt{\frac{0.8(0.2)}{n}}$$

$$0.0004 = \frac{0.16}{n}$$

$$\therefore n = \frac{0.16}{0.0004} = \frac{0.16 \times 10000}{0.0004 \times 10000} = \frac{1600}{4}$$

$$\therefore n = 400$$

Section E continued

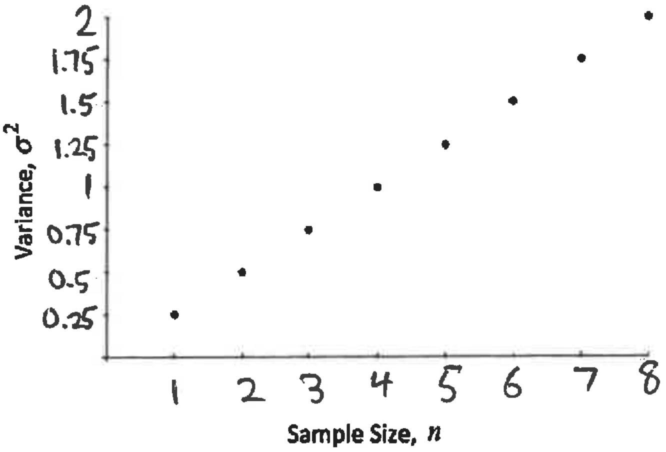
Marker use

Question 18

Variances for two binomial distributions are shown below.

a) Label the scales on both axes for each graph.

i. $X \sim Bi\left(n, \frac{1}{2}\right)$ for $n \in [1, 8]$



$\sigma^2 = np(1-p)$

$\sigma^2 = n\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right)$

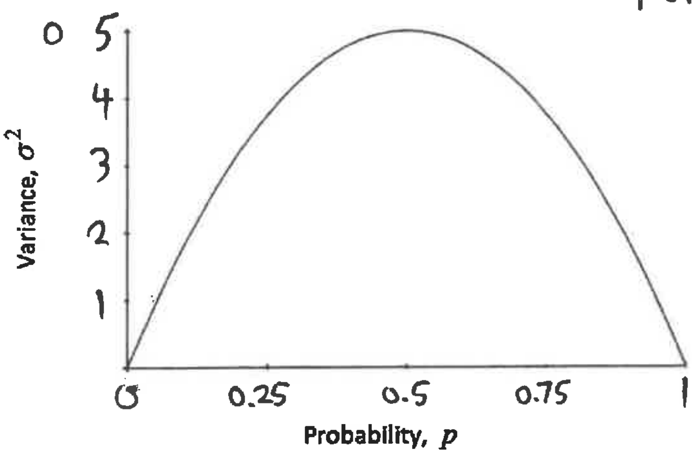
$\sigma^2 = \frac{n}{4}$

/2

ii. $Y \sim Bi(20, p)$

For $n = 20$, $\sigma^2 = \frac{20}{4} = 5$ when $p = \frac{1}{2}$

/2



Spare diagram used (✓)

b) Explain why points are plotted for graph i. whilst graph ii. is a continuous curve.

The possibilities for graph (i) are discrete, i.e. you either get 1 or 2 or 3 etc. up to 8 $\exists$ points

The possibilities for graph (ii) are continuous as p can vary anywhere from 0 to 1 inclusive as p merely measures the probability of success of a Bernouilli trial.

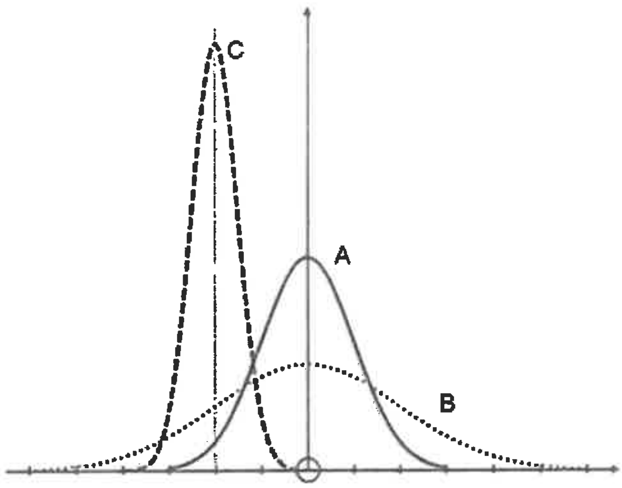
/2

Total C8

/16

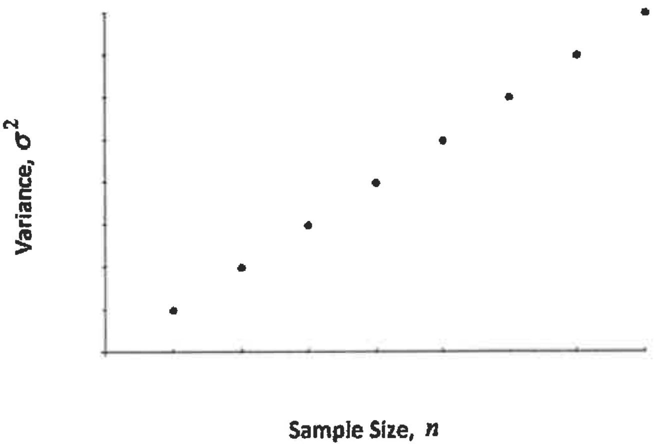
Spare Diagrams

Question 16

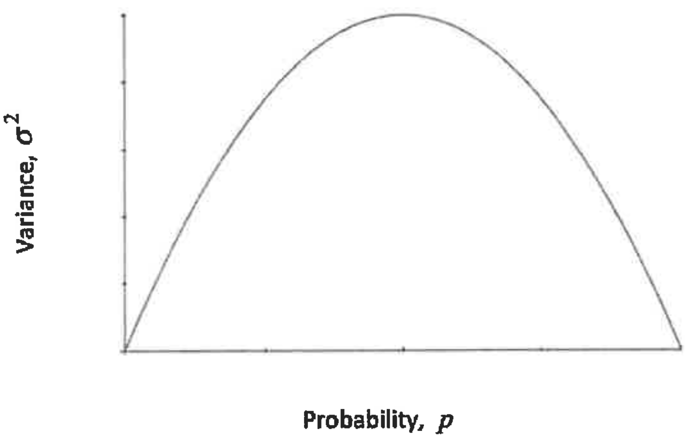


Question 18

i. $\mathrm{X} \sim \mathrm{Bi}\left(n, \frac{1}{2}\right)$ for $n \in [1,8]$



ii. $\mathrm{Y} \sim \mathrm{Bi}(20, p)$



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End of Part 1

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SOLUTIONS (continued)

MATHEMATICS METHODS

MTM415117

Part 2

Pages	28
Questions	22
Information Sheet	1

Suggested working time: 100 minutes

Instructions

Calculators are allowed to be used for Part 2.

- There are **five (5)** sections to this exam paper.
- Answer **all** questions and **all** parts within each question.
- Write your answers in the spaces provided in this exam paper.
  - Spare diagrams have been provided at the end of each section. Indicate in the box provided if you have used the spare diagram.
- During the first 80 minutes you may move onto Part 2, but you **cannot** use your calculator until told by your Supervisor(s).
- All answers must be written in **English**.
- You **must** make sure your answers address:
  - Criterion 4 understand polynomial, hyperbolic, exponential and logarithmic functions.
  - Criterion 5 understand circular functions.
  - Criterion 6 use differential calculus in the study of functions.
  - Criterion 7 use integral calculus in the study of functions.
  - Criterion 8 understand binomial and normal probability distributions and statistical inference.

Marker use	
	20
	20
	20
	20
	20

Additional Exam Instructions

For questions worth **one (1)** mark, you are not required to show workings. Markers will look at the presentation of the answer(s) and at the argument(s) leading to the final answer(s).

For questions worth **two (2)** or more marks **you are required** to show relevant workings.

Marks will be allocated:

- according to the degree to which workings convey a logical line of reasoning
- for suitable justifications and explanations of methods and processes when requested.

Guide to Exam Structure

	Sections	Questions available	How many questions to answer	Suggested working time	Marks available
Part 1	Section A	4	4	16 minutes	16
	Section B	3	3	16 minutes	16
	Section C	4	4	16 minutes	16
	Section D	3	3	16 minutes	16
	Section E	4	4	16 minutes	16
Total		18	18	80 minutes	80
Part 2	Section A	4	4	20 minutes	20
	Section B	4	4	20 minutes	20
	Section C	5	5	20 minutes	20
	Section D	5	5	20 minutes	20
	Section E	4	4	20 minutes	20
Total		**22**	**22**	**100 minutes**	**100**
Total		**40**	**40**	**180 minutes (3 hours)**	**180**

Section A

- Answer **all** questions in this section.
- This section assesses **Criterion 4.**

Question 19

Marker use

a) Determine the equation of the inverse for $f(x)=2\sqrt{x-1}$.

To find inverse: swap x & y in $y=2\sqrt{x-1}$

$\therefore x = 2\sqrt{y-1}$

$\frac{x}{2} = \sqrt{y-1}$

$\frac{x^2}{4} = y-1$

$y = \frac{x^2}{4}+1$

$\therefore f^{-1}(x) = \frac{x^2}{4}+1$

/2

b) Complete the table below for the domain and range of $f(x)$ and $f^{-1}(x)$.

	$f(x)$	$f^{-1}(x)$
Domain	$x \in [1, \infty)$	$x \in [0, \infty)$
Range	$y \in [0, \infty)$	$y \in [1, \infty)$

/2

Section A continues

Section A continued

Marker use

Question 20

a) Apply log laws to transform $f(x)=\log_2(x-1)-\log_2(x-1)^3+\log_2 4$

into the form $f(x)=a\log_2(x-1)+k$.

$f(x) = \log_2(x-1) - \log_2(x-1)^3 + \log_2 4$ where $x>1$

$= \log_2\left(\frac{(x-1)}{(x-1)^3}\right) + 2$

$= \log_2\left((x-1)^{-2}\right) + 2$

$= -2\log_2(x-1) + 2$

/2

b) If $x=\log_a b$, $y=\log_b c$ and $z=\log_c a$, use the change of base theorem to show that $\underbrace{xyz}_{LHS}=\underbrace{1}_{RHS}$.

$LHS = (\log_a b)(\log_b c)(\log_c a)$

$= (\cancel{\log_a b})\left(\frac{\cancel{\log_a c}}{\cancel{\log_a b}}\right)\left(\frac{\log_a a}{\cancel{\log_a c}}\right)$

$= \log_a a$

$= 1$

$= RHS$

/2

Section A continues

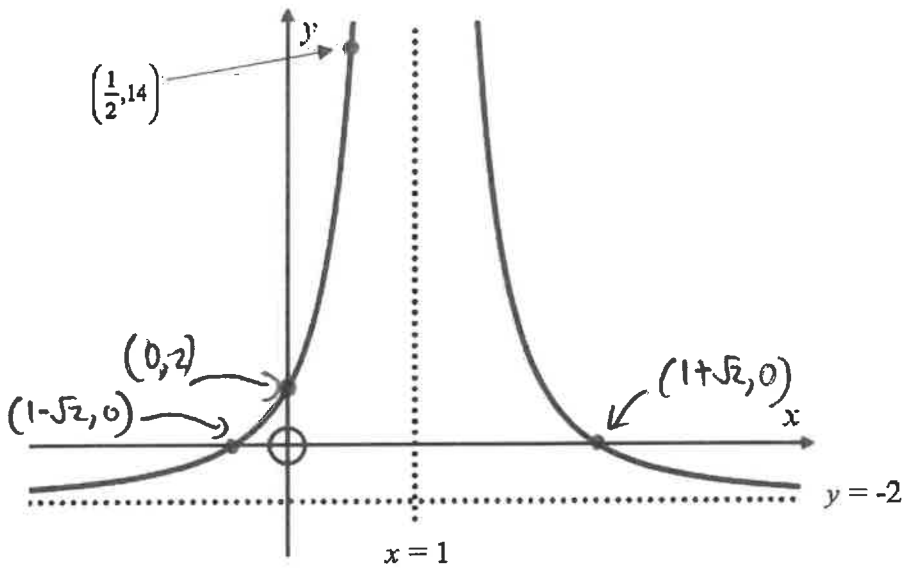
Section A continued

Marker use

/ 4

Question 21

Find and label, with exact values, the x and y intercepts of the truncus graphed below.



Spare diagram used (✓) ☐

Truncus has general equation $y = \frac{a}{(x-1)^2} - 2$

using asymptotes.

Using $(\frac{1}{2}, 14)$: $14 = \frac{a}{(\frac{1}{2}-1)^2} - 2$

$16 = \frac{a}{\frac{1}{4}}$

$\therefore a = 4$

$\therefore$ Equation is $y = \frac{4}{(x-1)^2} - 2$

y-intercept: when $x=0 \Rightarrow y = \frac{4}{(-1)^2} - 2 = 2$

x-intercepts: when $y=0 \Rightarrow 0 = \frac{4}{(x-1)^2} - 2$

$2 = \frac{4}{(x-1)^2}$

$(x-1)^2 = 2$

$x-1 = \pm\sqrt{2}$

$x = 1 \pm \sqrt{2}$

Section A continues

Marker use

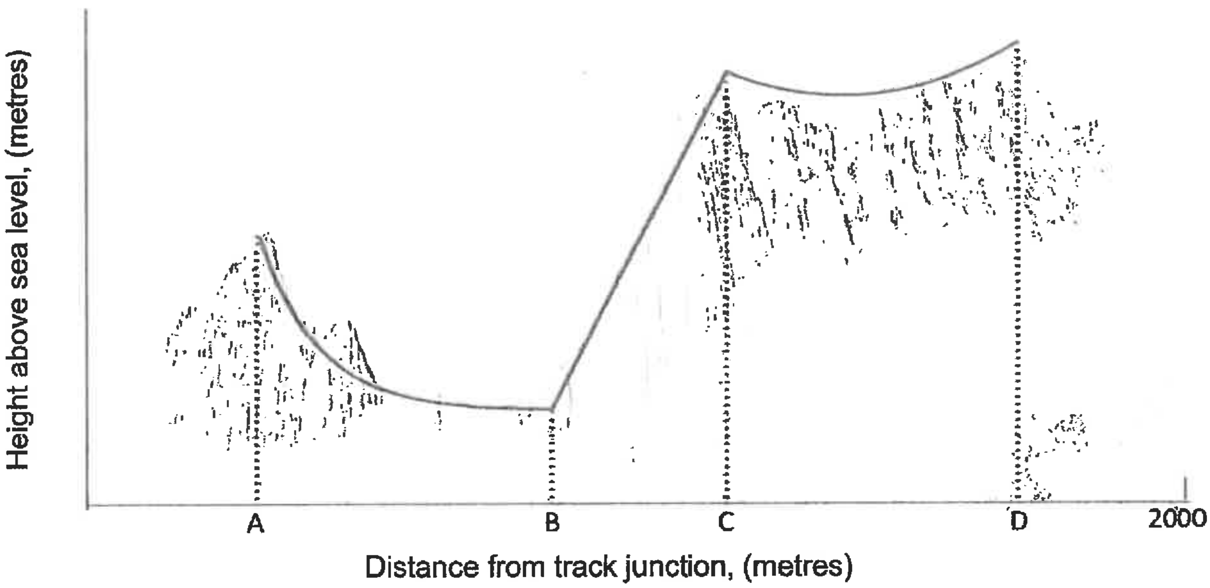
Question 22

A profile of Cradle Mountain is graphed below and modelled by the functions $f(x)$, $g(x)$ and $h(x)$ where:

$$f(x) = e^{\frac{-1}{100}(x-800)} + 1230 \quad \text{for} \quad x \in [\mathrm{A}, \mathrm{B}]$$

$$g(x) = \frac{289x}{300} + \frac{1381}{3} \quad \text{for} \quad x \in [\mathrm{B}, \mathrm{C}]$$

$$h(x) = \frac{x^2}{2000} - \frac{13x}{10} + 2345 \quad \text{for} \quad x \in [\mathrm{C}, \mathrm{D}]$$



The x coordinates are horizontal distances from a track junction and the y coordinates represent heights above sea level.

Units are in metres. The diagram above is not to scale.

Question 22 continues

Question 22 continued

Marker use

a) State the four graphical transformations from $y = e^x$ to $f(x) = e^{\frac{-1}{100}(x-800)} + 1230$.

- Reflection in the y axis
- Dilation by factor 100 in the x direction
- Translation $\xrightarrow[800m]{}$ (to the right) and $\uparrow$ 1230m (up).

/3

b) Smithies Peak occurs when $x = \mathrm{C}$. Cradle Peak is the highest at 1545m and occurs when. $x = \mathrm{D}$. Determine the **difference in distance** from the track junction and the **height variation** between the Smithies and Cradle Peaks.

(State relevant equations and reasons for validating any solutions.)

/5

Smithies Peak will occur when $g(x) = h(x)$

$$\frac{289x}{300} + \frac{1381}{3} = \frac{x^2}{2000} - \frac{13x}{10} + 2345$$

Solving via CAS gives $x = 1100$ or $x = \frac{10280}{3} = 3426.\dot{6}$

Only solution within the domain $0 < x < 2000$ (graph scale) is $x = 1100$

When $x = 1100$, height of Smithies Peak $= \frac{289(1100)}{300} + \frac{1381}{3}$

$= 1520$ metres

When height $= 1545$ (Cradle Peak height), x is obtained by solving $1545 = \frac{x^2}{2000} - \frac{13x}{10} + 2345$

giving $x = 1000, 1600$

$\therefore$ Only solution > 1100 (beyond Smithies Peak) is $x = 1600$

$\therefore$ Difference in distance from track junction $= 1600 - 1100 = 500$ metres

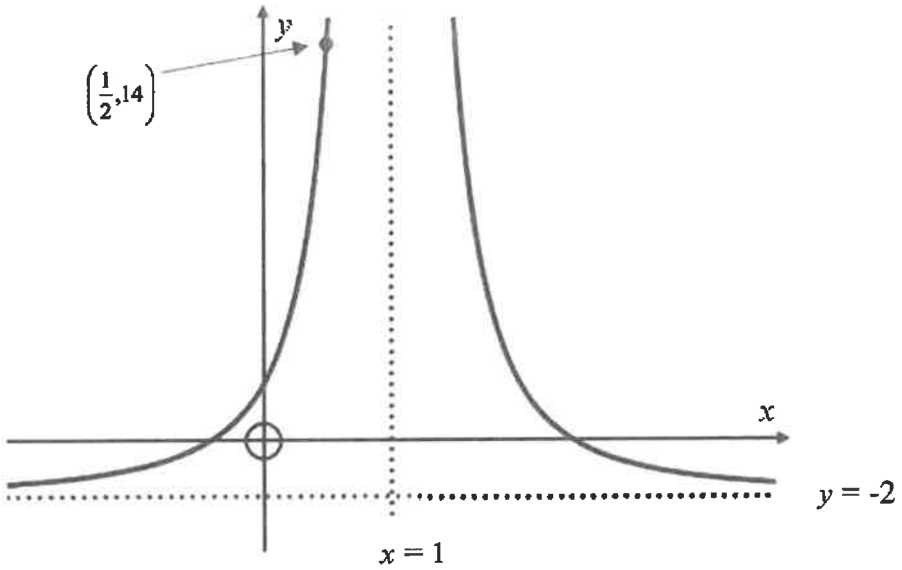
Height variation between Smithies and Cradle Peaks $= 1545 - 1520 = 25$ metres

Total C4

/20

Spare Diagram

Question 21

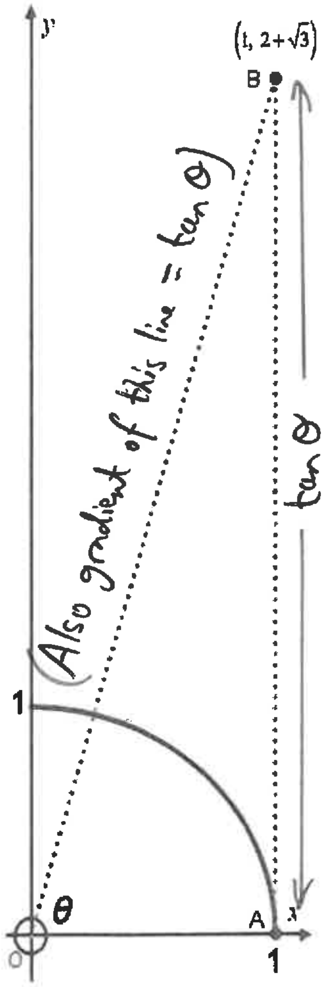


Section B

- Answer **all** questions in this section.
- This section assesses **Criterion 5**.

Question 23

Marker use



The first quadrant of a unit circle and the right-angled triangle AOB are shown to the left.

The triangle makes an angle θ at the origin and the coordinate of point B is $\left(1,\ 2+\sqrt{3}\right)$.

a) Label $\tan\theta$ on the diagram. /1

b) State and solve a trigonometric equation to determine an exact value for θ. /2

$\tan\theta = 2+\sqrt{3}$ for $0<\theta<\frac{\pi}{2}$

$\therefore \theta = \tan^{-1}(2+\sqrt{3})$

$= \frac{5\pi}{12}$ via CAS

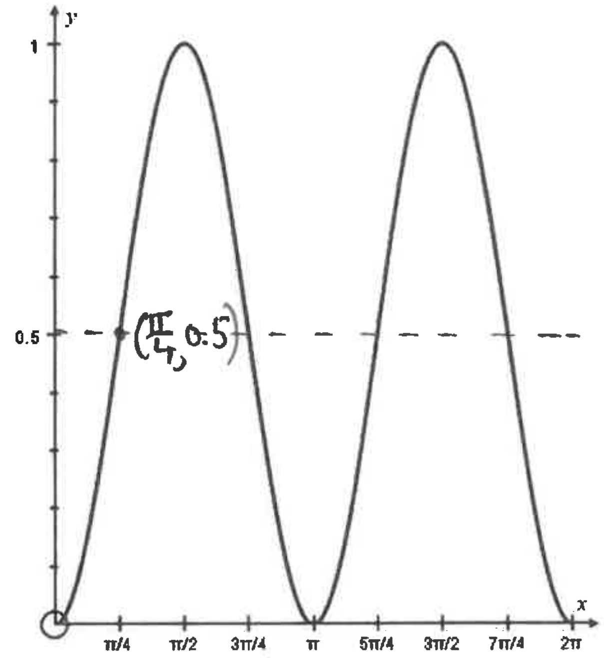
Spare diagram used (✓)

☐

Section B continues

Section B continued

Question 24



For the graph shown above:

a) determine a cosine function of the form $f(x) = a\cos(bx) + c$.

Period $= \frac{2\pi}{b} = \pi$ so $b = 2$

Function is cos reflected in the x axis and then translated upwards

$\therefore$ c = midline $= 0.5$

a = amplitude $= 0.5$

No horizontal translations

$\therefore f(x) = -0.5\cos(2x) + 0.5$

3

b) determine a sine function of the form $g(x) = d\sin[e(x - f)] + g$.

Function is also a sin function translated $\frac{\pi}{4}$ to the right

Same period, amplitude and midline

$\therefore$ Possible $g(x) = 0.5\sin(2(x - \frac{\pi}{4})) + 0.5$

Other possible sine functions are : $g(x) = -0.5\sin(2(x + \frac{\pi}{4})) + 0.5$

or $g(x) = -0.5\sin(2(x - \frac{3\pi}{4})) + 0.5$

3

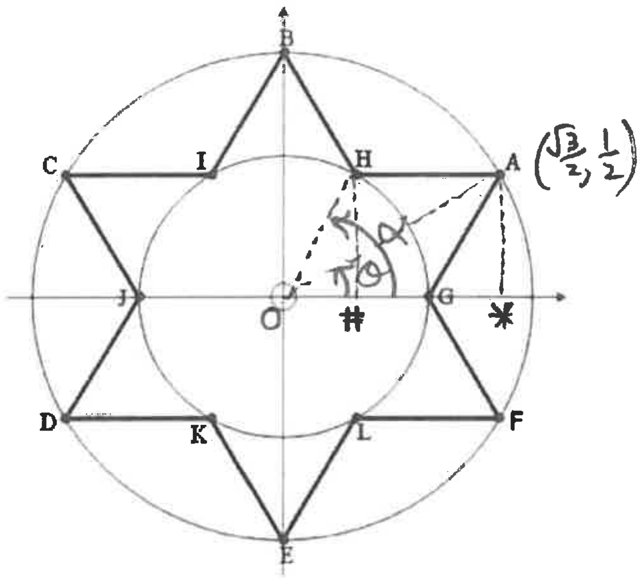
Marker use

Section B continued

Marker use

Question 25

The star shown to the right has six outer points (A to F) equally spaced around a **unit circle**.



a) Find the co-ordinates of point A.

Due to six equally spaced points A to F and symmetry about the x axis, $\theta = \frac{\pi}{6}$

By definition of the unit circle, A is at $\left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right)$

$= \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

/3

Six inner points (G to L), evenly distanced from the closest outer points, lie on a circle with a smaller radius. The line $\overline{AH}$ is parallel to the x axis.

b) Show the inner circle radius is $\frac{\sqrt{3}}{3}$.

By symmetry, $\alpha = \frac{\pi}{3}$

Line $\overline{OH}$ is given by $y = \tan\left(\frac{\pi}{3}\right)x$

$y = \sqrt{3}x$

Line $\overline{AH}$ is given by $y = \frac{1}{2}$

Alternative solution:

Consider triangle HO#

By symmetry $\angle HO\# = \frac{\pi}{3}$

$\therefore \sin\frac{\pi}{3} = \frac{opp}{hyp} = \frac{\frac{1}{2}}{\text{inner radius}}$

$\therefore \frac{\sqrt{3}}{2} = \frac{\frac{1}{2}}{\text{inner radius}}$

So inner radius $= \frac{1}{2} \times \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$\therefore$ H has coordinates $\left(\frac{1}{2\sqrt{3}}, \frac{1}{2}\right)$

$\therefore \overline{OH} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2\sqrt{3}}\right)^2} = \sqrt{\frac{1}{4} + \frac{1}{12}} = \sqrt{\frac{4}{12}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

/3

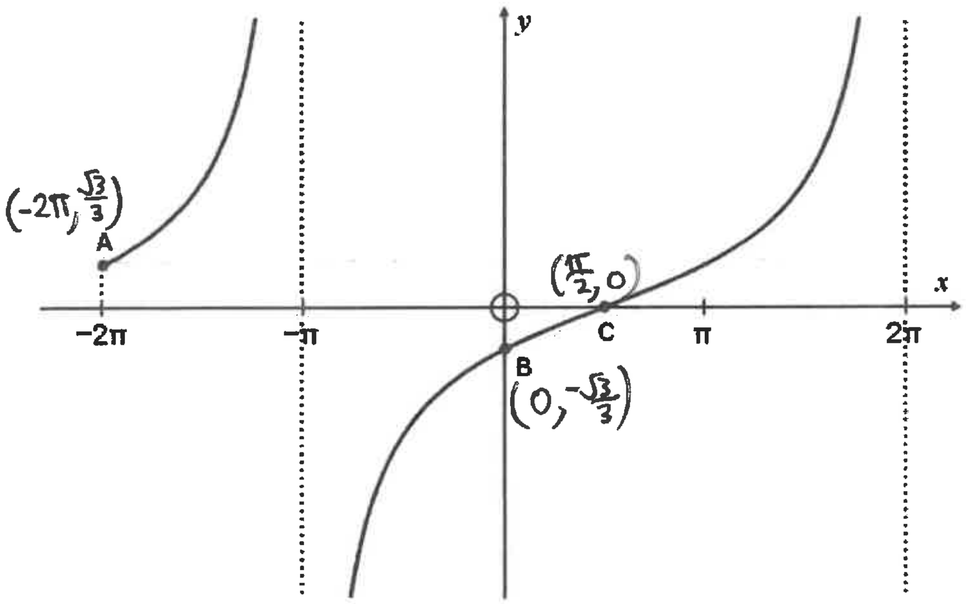
Section B continues

Marker use

Question 26

A function of the form $y = \tan n(x+b)$ is shown below with vertical asymptotes at $x = -\pi$ and 2π.

$\Rightarrow$ no vertical translation



is also the midline

Determine the coordinates for the end point, A and the axes intercepts for B and C.

/5

Midline intercept half-way between vertical asymptotes

$\therefore$ C's x coordinate is $\frac{-\pi+2\pi}{2} = \frac{\pi}{2}$ $\therefore b = -\frac{\pi}{2}$

Period is $2\pi - (-\pi) = 3\pi \Rightarrow n = \frac{\pi}{3\pi} = \frac{1}{3}$

$\therefore$ Equation is $y = \tan\left(\frac{1}{3}\left(x - \frac{\pi}{2}\right)\right)$

y-intercept when $x = 0 \Rightarrow y = \tan\left(-\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{3}$ (angle in 4th quadrant)

$\therefore$ B is at $\left(0, -\frac{\sqrt{3}}{3}\right)$

C is at $\left(\frac{\pi}{2}, 0\right)$

A is at $\left(-2\pi, \tan\left(\frac{1}{3}\left(-2\pi - \frac{\pi}{2}\right)\right)\right)$

$= \left(-2\pi, \tan\left(-\frac{5\pi}{6}\right)\right)$

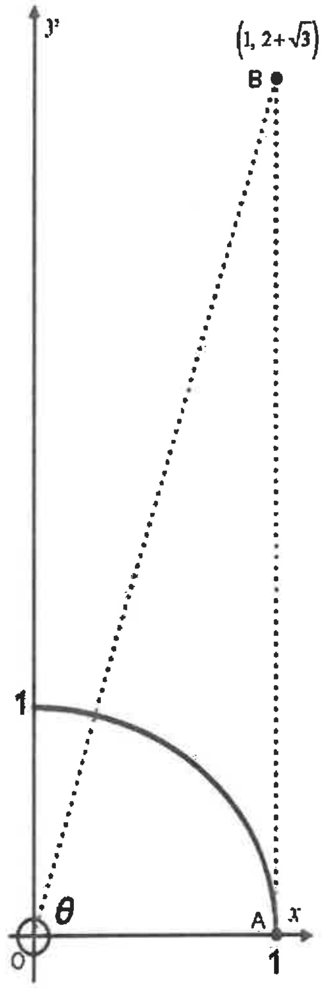
$= \left(-2\pi, \frac{\sqrt{3}}{3}\right)$

Total C5

/20

Spare Diagram

Question 23



Section C

- Answer **all** questions in this section.
- This section assesses **Criterion 6**.

Question 27

Marker use

$f(x) = 2ax^2 - bx$ where a and b are constants.

a) Determine an expression for the instantaneous rate of change at $x = 2$.

/2

$f'(x) = 2a(2x) - b(1) = 4ax - b$

inst. rate of change at $x=2$ is $f'(2) = 4a(2) - b$

$= 8a - b$

b) Show that the average rate of change between $x = 1$ and $x = 3$ is the same as the instantaneous rate of change in part a).

/2

average rate of change from $x=1$ to $x=3$

$= \frac{f(3) - f(1)}{3-1}$

$= \frac{2a(3)^2 - 3b - 2a(1)^2 + b}{2}$

$= \frac{18a - 3b - 2a + b}{2}$

$= \frac{16a - 2b}{2}$

$= 8a - b$

$= f'(2)$ in this case

(but not in the general case for any expression for $f'(x)$)

Section C continues

Marker use

Question 28

Find the x coordinate(s) where $f(x)=\ln(2x)$ and $g(x)=\dfrac{-9}{2x^2}$ have the same gradient.

/ 3

$f'(x) = \dfrac{d\ln(2x)}{d(2x)} \cdot \dfrac{d(2x)}{dx} = \dfrac{2}{2x} = \dfrac{1}{x}$

$g'(x) = -\frac{9}{2}x^{-2} = -\frac{9}{2}(-2)x^{-3} = \dfrac{9}{x^3}$

The functions have the same gradient for $f'(x) = g'(x)$

$\dfrac{1}{x} = \dfrac{9}{x^3}$

$1 = \dfrac{9}{x^2}$ $\quad x \neq 0$

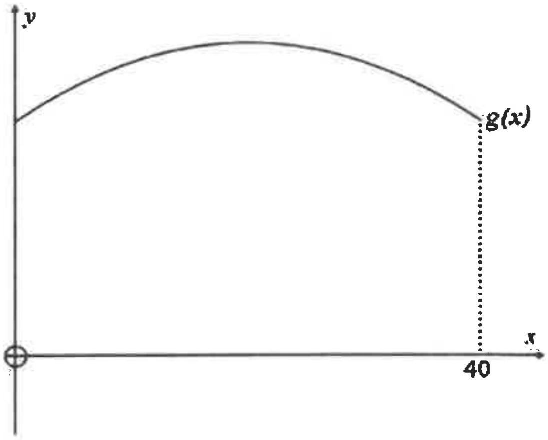
$x^2 = 9$

$x = \pm 3$

But reject $x=-3$ as $f(x)$ is undefined so $x=3$ is the only solution.

Question 29

$g(x)=\dfrac{-x^2}{400}+\dfrac{x}{10}+3$ models the walkway of a bridge that spans 40m.



Determine all values of x where the **magnitude** of the gradient is 0.06 or less.

/ 4

$g'(x) = \dfrac{-2x}{400} + \dfrac{1}{10} = \dfrac{-x}{200} + \dfrac{1}{10}$

We require $\left| \dfrac{-x}{200} + \dfrac{1}{10} \right| \leq 0.06$

i.e. $-0.06 \leq \dfrac{-x+20}{200} \leq 0.06$

$-12 \leq -x+20 \leq 12$

$-32 \leq -x \leq -8$

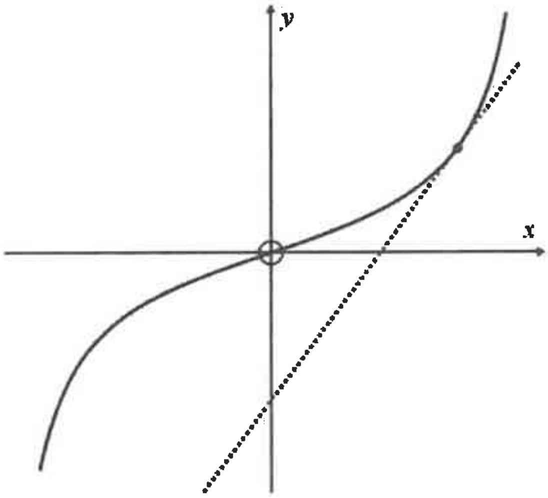
$32 \geq x \geq 8$

$\therefore$ We require $8 \leq x \leq 32$, i.e. $x \in [8, 32]$

Marker use

Question 30

A graph of $y = \tan(2x)$ and its tangent at $x = \dfrac{\pi}{6}$ are shown below.



Determine the equation of the tangent.

$\frac{dy}{dx} = \frac{d\tan(2x)}{d(2x)} \cdot \frac{d(2x)}{2} = 2\sec^2(2x)$

$\therefore$ At $x = \frac{\pi}{6}$, $m_{tangent} = \frac{2}{\cos^2(\frac{2\pi}{6})} = \frac{2}{(\frac{1}{2})^2} = 8$

When $x_1 = \frac{\pi}{6}$, $y_1 = \tan(\frac{2\pi}{6}) = \tan(\frac{\pi}{3}) = \sqrt{3}$

$\therefore$ Equation of tangent is $y - y_1 = m_{tangent}(x - x_1)$

$y - \sqrt{3} = 8(x - \frac{\pi}{6})$

$y = 8x - \frac{8\pi}{6} + \sqrt{3}$

$y = 8x - \frac{4\pi}{3} + \sqrt{3}$

/4

Section C continues

Section C continued

Marker use

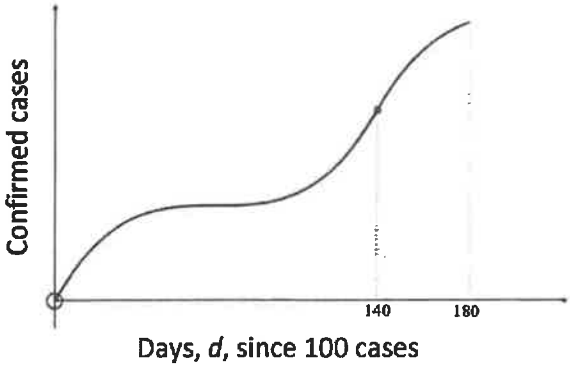
Question 31

The graph shown models the number of confirmed cases of a virus, $C(d)$, d days since the first 100 cases were recorded in a country.

The graph of $C(d)$ is a combination of the two functions $f(d)$ and $g(d)$ where:

$$f(d)=\frac{1}{50}(d-70)^3+6860\,,\quad d\in[0,\,140]$$

$$g(d)=\frac{1}{50}(d-210)^3+20580\,,\quad d\in[140,\,180]$$



a) Show that $C(d)$ is **differentiable** at $d=140$ — 4

$C(d)$ is differentiable at $d=140$ if C is continuous & smooth at $d=140$

$\therefore$ We require $\underbrace{f'(140)=g'(140)}_{\text{smooth}}$ and $\underbrace{f(140)=g(140)}_{\text{continuous}}$

$f'(d)=\frac{3}{50}(d-70)^2$ so $f'(140)=\frac{3}{50}(140-70)^2=\frac{3(70)^2}{50}=294$

$g'(d)=\frac{3}{50}(d-210)^2$ so $g'(140)=\frac{3}{50}(140-210)^2=\frac{3(-70)^2}{50}$

$=\frac{3(70)^2}{50}=294$

Thus $f'(140)=g'(140)$ (same gradients from each piece-wise component over joining point)

$f(140)=\frac{1}{50}(140-70)^3+6860=13720$

$g(140)=\frac{1}{50}(140-210)^3+20580=13720$

Thus $f(140)=g(140)$. (f & g join continuously at $d=140$)

b) Explain the meaning of $C'(140)$. — 1

$C'(140)$ is the gradient of the tangent to $C(d)$ which is defined as C is smooth and continuous at $d=140$

Also, $C'(140)$ is the rate of change of cases at the 140th day after the first 100 cases were recorded.

Total C6 — 20

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Exam continues over the page

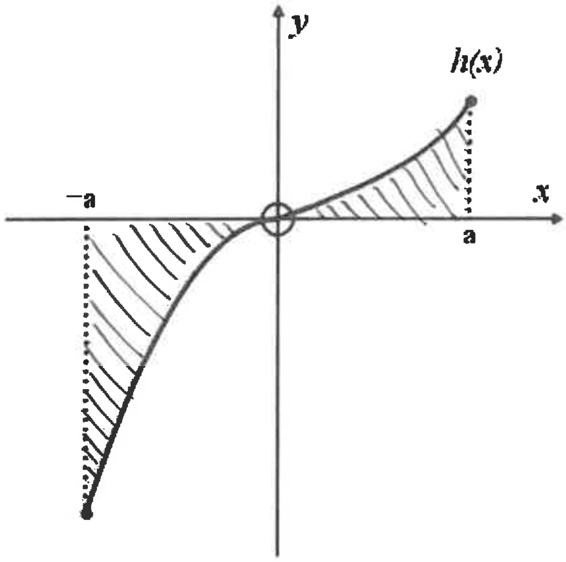
Section D

- Answer **all** questions in this section.
- This section assesses **Criterion 7**.

Question 32

Marker use

The function $h(x)$ is graphed to the right.



a) Write an expression to determine the shaded area.

Shaded area $= -\int_{-a}^{0} h(x)\,dx + \int_{0}^{a} h(x)\,dx$

(Some other ways this can be written: shaded area $= \int_{0}^{-a} h(x)\,dx + \int_{0}^{a} h(x)\,dx$

shaded area $= \left|\int_{-a}^{0} h(x)\,dx\right| + \int_{0}^{a} h(x)\,dx$.)

/2

b) Explain the meaning of $\int_{-a}^{a} h(x)\,dx = -1$ in terms of the shaded area above.

The shaded area from $x=-a$ to $x=0$ is 1 unit larger than the shaded area from $x=0$ to $x=a$ which means that the integral is negative from $x=-a$ to $x=a$ since the integral of a function below the axis gives a negative value (assuming the bottom terminal is smaller than the top terminal).

/2

Section D continues

Marker use

Question 33

Evaluate $\int_a^b f(x)\mathrm{d}x$ given:

$$\int_a^b f(x)\mathrm{d}x - \int_1^4 2\mathrm{d}x = \int_b^a f(x)\mathrm{d}x.$$

$$\int_a^b f(x)\,dx - \int_1^4 2\,dx = -\int_a^b f(x)\,dx$$

$$2\int_a^b f(x)\,dx = \int_1^4 2\,dx$$

$$\int_a^b f(x)\,dx = \tfrac{1}{2}\left[2x\right]_1^4$$

$$\int_a^b f(x)\,dx = \tfrac{1}{2}\left(2(4)-2(1)\right)$$

$$= \tfrac{1}{2}(6) = 3$$

/ 3

Question 34

An object moves from the start in a straight line with a velocity given by:

$$v = \cos\left(\frac{\pi t}{2}\right) + 4t$$

where v is in metres/second and t is in seconds.

Provide calculus reasoning to show the distance covered by the object in the first 4 seconds is 32 metres.

$$s = \int_0^4 v\,dt$$

$$= \int_0^4 \left(\cos\left(\tfrac{\pi t}{2}\right) + 4t\right) dt$$

$$= \left[\frac{\sin\left(\frac{\pi t}{2}\right)}{\pi/2} + \frac{4t^2}{2}\right]_0^4$$

$$= \left[\frac{2\sin\left(\pi t/2\right)}{\pi} + 2t^2\right]_0^4$$

$$= \left(\frac{2\sin(2\pi)}{\pi} + 2(4)^2\right) - \left(\frac{2\sin(0)}{\pi} + 2(0)^2\right)$$

$$= 0 + 32 - 0 - 0 = 32 \text{ metres}$$

/ 3

Section D continues

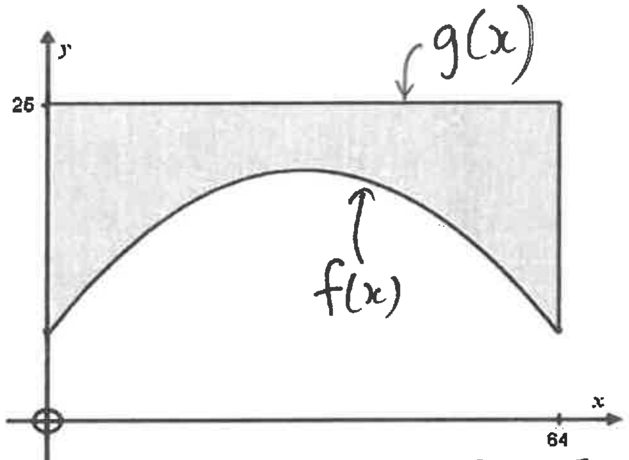
Marker use

Question 35

An arch and road surface 64 metres long are modelled by the functions:

$$f(x) = 170\sin\left(\frac{\pi}{256}(x+96)\right) - 150 \quad \text{and} \quad g(x) = 25$$

The graph below shows the cross-sectional area bounded by the arch and road.



$$g(x) - f(x) = 25 - \left(170\sin\left(\tfrac{\pi}{256}(x+96)\right) - 150\right)$$
$$= 175 - 170\sin\left(\tfrac{\pi}{256}(x+96)\right)$$

a) Write an integral expression to find the cross-sectional area.
Determine this area accurate to one decimal place. /3

Cross-sectional area $= \int_0^{64} (g(x) - f(x))\,dx$

$= \int_0^{64} \left(175 - 170\sin\left(\tfrac{\pi}{256}(x+96)\right)\right) dx$

$= \left[175x - \dfrac{170\cos\left(\frac{\pi}{256}(x+96)\right)}{\pi/256}\right]_0^{64}$

$= \left(175(64) - \dfrac{256(170)\cos\left(\frac{\pi}{256}(64+96)\right)}{\pi}\right) - \left(0 - \dfrac{256(170)\cos\left(\frac{96\pi}{256}\right)}{\pi}\right)$

$= 597.5\ m^2$ (to 1 decimal place)

b) The width of the road is 12 metres. Calculate the volume of rock required to fill the space between the arch and road surface. /1

Volume = cross-sectional area × width

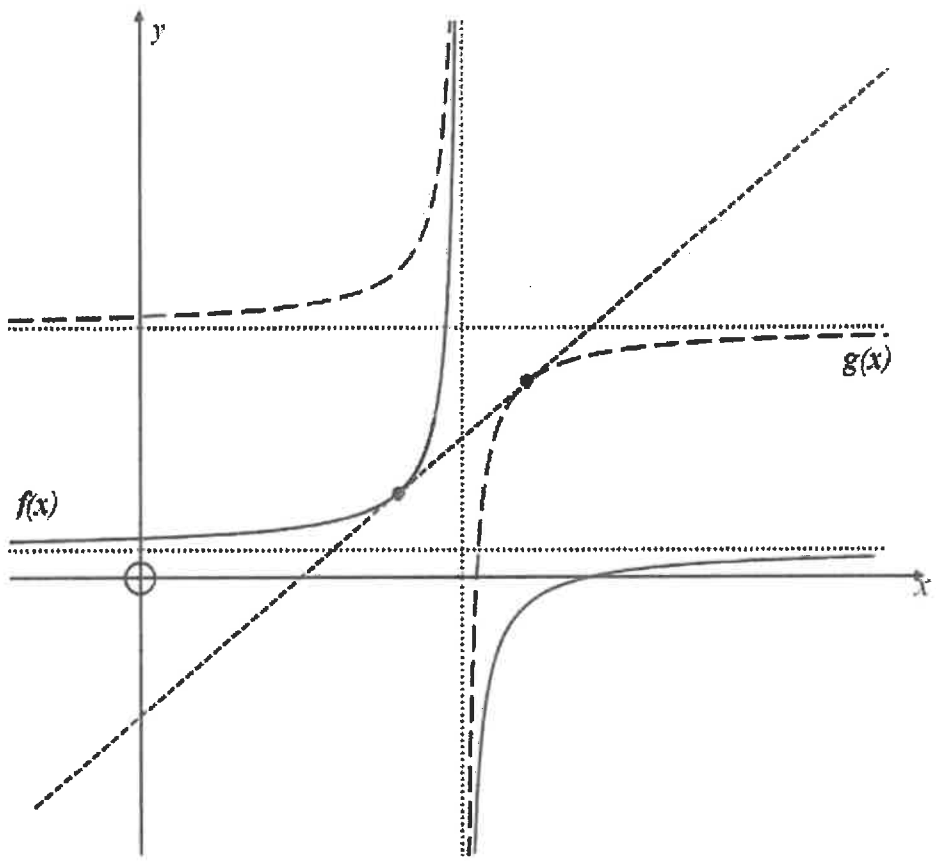
$= 597.5 \times 12 = 7170\ m^3$

Marker use

Question 36

Two hyperbolas, $f(x)$ and $g(x)$ have the same gradient function equal to $\dfrac{4}{(2x-5)^2}$.

The line $4x-y-5=0$ is a tangent to both functions and intersects at two points as shown in the graph below.



Question 36 continues

Marker use

Determine the equations for $f(x)$ and $g(x)$.

The line $4x - y - 5 = 0$

$y = 4x - 5$ has a gradient of 4

$f(x)$ and $g(x)$ have same gradient function

$\Rightarrow g(x)$ is $f(x)$ translated up or down

since $f(x)$ and $g(x)$ differ only by a constant

$f(x) = \int f'(x)\,dx = \int 4(2x-5)^{-2}\,dx = \frac{4(2x-5)^{-1}}{(2)(-1)} + c_1$

$= -2(2x-5)^{-1} + c_1$ where c_1 is a constant

$g(x) = \int g'(x)\,dx = -2(2x-5)^{-1} + c_2$ where c_2 is a constant

x coordinates of intersection points given by $f'(x) = 4$

$\frac{4}{(2x-5)^2} = 4$

$(2x-5)^2 = 1$

$2x - 5 = \pm 1$

$x = 2$ or 3

The diagram shows that intersection between $f(x)$ and line is at $x=2$

$\Rightarrow y = 4(2) - 5 = 3$

$\therefore f(2) = 3 \Rightarrow \frac{-2}{2(2)-5} + c_1 = 3$

$2 + c_1 = 3 \Rightarrow c_1 = 1$

Similarly intersection between $g(x)$ and line is at $x = 3$

$\Rightarrow y = 4(3) - 5 = 7$

$\therefore g(3) = 7 \Rightarrow \frac{-2}{2(3)-5} + c_2 = 7$

$-2 + c_2 = 7 \Rightarrow c_2 = 9$

$\therefore f(x) = \frac{-2}{2x-5} + 1$ and $g(x) = \frac{-2}{2x-5} + 9$

/6

Total C7

/20

Section E

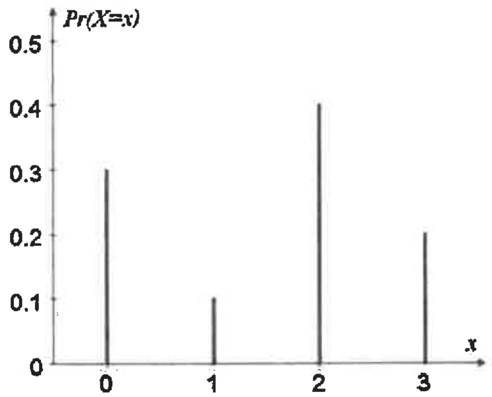
- Answer **all** questions in this section.
- This section assesses **Criterion 8**.

Marker use

Question 37

The probability distribution for the discrete random variable X is graphed below. Determine the expected value and standard deviation for this distribution.

/3



$E(X) = \sum x \Pr(X=x) = 0(0.3) + 1(0.1) + 2(0.4) + 3(0.2)$

$= 1.5$

$E(X^2) = 0^2(0.3) + 1^2(0.1) + 2^2(0.4) + 3^2(0.2)$

$= 0.1 + 1.6 + 1.8 = 3.5$ or

$SD(X) = \sqrt{Var(X)} = \sqrt{E(X^2) - (E(X))^2} = \sqrt{3.5 - 1.5^2} = \sqrt{1.25} = 1.12$

(to 2 decimal places)

or $\sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$

Question 38

Table tennis balls are required to meet a "40+" standard which sets allowable diameters between 40 and 40.2 mm.

It is thought that about 99% of a premium ball meets the "40+" requirement.

Determine the smallest sample needed to establish this to within a margin of error of 3% at a 95% confidence interval.

/4

$M = 0.03$, $z = 1.96$ for a 95% confidence level, $\hat{p} = 0.99$

$\therefore\ 0.03 = 1.96\sqrt{\frac{0.99(1-0.99)}{n}}$

Solving via CAS gives $n = 42.26$

$\therefore$ Minimum sample size needed $= 43$ (rounding up)

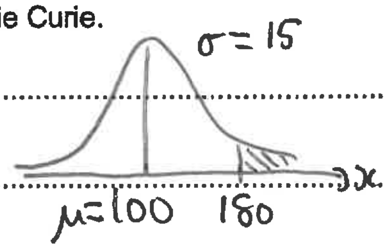
Section E continues

Question 39

IQ test results are normally distributed with a mean of 100 and a standard deviation of 15.

a) Marie Curie reportedly had an IQ of 180. Given Australia has a population of about 26 million, determine how many Australians are likely to have an IQ greater than Marie Curie.

/3



Let $X \sim N(\mu = 100, \sigma = 15)$

$\Pr(X > 180) = 4.821 \times 10^{-8}$

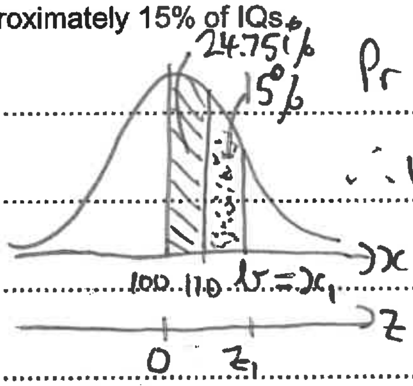
$\therefore$ For 26 million people, number of Australians $= 26 \times 10^{6} \times 4.821 \times 10^{-8}$

$= 1.25$

So most likely number of people $= 1$ (rounding to the nearest integer)

b) Determine the value of b, to the nearest integer, such that $\Pr(110 < x < b)$ represents approximately 15% of IQs.

/4



$\Pr(100 < X < 110) = 0.24751$ or 24.751%

$\therefore$ We require $\Pr(100 < X < b) = 0.24751 + 0.15$

$= 0.39751$

or $\Pr(X > b) = 0.5 - 0.39751$

$= 0.10249$

Now $\Pr(Z > z_1) = 0.10249$ for $Z \sim N(\mu = 0, \sigma = 1)$

gives $z_1 = 1.26749$

Transforming back to our "X" normal distribution

$z_1 = \frac{x_1 - \mu}{\sigma}$

$1.26749 = \frac{b - 100}{15}$

$\therefore b = 15(1.26749) + 100$

$= 119.01$

$\therefore$ To the nearest integer, $b = 119$

Section E continues

Marker use

Marker use

Question 40

a) Given $X \sim Bi(15, p)$ and $Pr(X = 0) = 0.02$, show p is approximately equal to 0.23 — /2

For a binomial distribution, $Pr(X=0) = {}^{15}C_0\, p^0 (1-p)^{15} = (1)(1)(1-p)^{15}$

$\therefore (1-p)^{15} = 0.02$

$\therefore 1 - p = (0.02)^{\frac{1}{15}}$

$\therefore p = 1 - (0.02)^{\frac{1}{15}} \simeq 0.2296 \simeq 0.23$

b)

i. A local restaurant seats 15 people and is fully booked for an extended period. On average, 23% of customers order a "gluten free" option. Determine the probability that three or more customers order "gluten free" on a given night. — /2

Let X = number of customers ordering the "gluten free" option

$X \sim Bi(n=15, p=0.23)$

$Pr(X \geq 3) = 0.7055$ or 70.55%

ii. The restaurant opens six nights a week. Determine the probability that three or more customers will order "gluten free" on 5 out of the 6 nights. — /2

Let Y = number of nights that 3 or more customers will order "gluten free"

$\therefore Y \sim Bi(n=6, p=0.7055)$

$Pr(Y=5) = 0.3088$ or 30.88%

Total C8 /20

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End of Part 2

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