2022 ASSESSMENT REPORT

MTM415117 - Mathematics Methods

GENERAL COMMENTS

Similar general comments apply to this year as in previous years. The exam contained questions aimed at a variety of skill levels and it was pleasing to see many students make a reasonable attempt at most questions. For the most part, students appeared to be able to identify what was being asked of them.

The exam featured numerous 'show that' questions which were intended as hints or to allow students access to subsequent items of a question if they were unable to approach the initial stages. However, this was often detrimental for those students who were not familiar with how to approach this style of question. Students needed to be confident about using the calculator appropriately in Section B, and equally to be secure in their algebra skills where required throughout the exam.

Students should be reminded to be as clear as possible in their written responses. Very small 0's and a's appeared unclear in some students' writing, especially when used as terminals of integral expressions. The same applies to incomplete bracketing. Some students write their square root symbol with a pronounced horizontal bar before the rest of the symbol. This makes it hard to determine if there is an intended minus sign in front of the horizontal bar.

It is good exam technique to make sure, even if you feel under time pressure, that you attempt all multiple-choice questions as you may get the question correct and receive appropriate marks as a reward even if you make a guess.

These were the threshold marks for a 'C', 'B' or 'A' as determined by the Assessment Panel. All marks are out of 36.

Criterion	Threshold mark for a 'C'	Threshold mark for a 'B'	Threshold mark for an 'A'
Criterion 4: Function Study	16.0	22.5	29.5
Criterion 5: Circular Functions	14.0	24.0	31.0
Criterion 6: Differential Calculus	12.5	21.5	29.5
Criterion 7: Integral Calculus	12.5	23.0	29.5
Criterion 8: Probability	12.0	23.5	32.0

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The exam will now be commented on question by question. Prospective students are strongly encouraged to read the comments in this assessment report, as well as reading the solutions to the exam questions.

FUNCTION STUDY (SECTION A)

QUESTION 1

(a) Students did very well on this question item. There were very few errors and most related to the shape of the hyperbola.

(b) Students also succeeded at answering this well. A few students used square brackets rather than round brackets or reversed the order of the domain's start or endpoints. Consideration was given for errors carried forward from item (a).

QUESTION 2

(a) This item was attempted with a high degree of success. Any mistakes were largely related to finding the value for a and often involved simple arithmetic errors.

(b) This item produced mainly correct answers with the most common error being either not including the endpoints in the range expression or instead using F as one of the endpoints of the range. Occasionally students gave an answer for a parabolic shape that was upside down.

(c) This item was completed very successfully by most students. Most of the errors were due to incorrect arithmetic in coming up with the y value of the point.

QUESTION 3

(a) Students needed to plot a sufficient number of points to produce a sufficiently accurate graph for item (b). While this question was often completed quite well, the most common errors related to the y- intercept value or the endpoint at $(2{,}8)$.

(b) Whether or not students gained full marks for this part was most often connected to the quality of their graph from item (a). Allowance was made for the fact that the curve was drawn by hand and a range of answers from 1.6 to 1.9 were accepted.

(c) This item proved challenging, with only a few students gaining full marks. Many students started with a productive approach by taking the natural logarithm of both sides of each equation and then applying relevant log laws, but the algebra rearrangement and the collection of 'like' x terms was often done poorly. An understanding that students may find useful is to realise that $\ln(2)$ is just a number and can be treated the same was as any integer or constant like π.

CIRCULAR FUNCTIONS (SECTION A)

QUESTION 4

(a) This item was completed very well, with many students following the correct conversion procedure. Arithmetic errors due to multiplication and division of two- and three-digit numbers were the main issue, so regular practice of these very basic skills may be useful.

(b) This item tended to be either well done or poorly done, with very few students scoring middle marks. Marks were awarded for correctly realising where the angles occurred in the unit circle and for correctly labelling (either as coordinates or lengths) the appropriate value of the ratios. However, the notation used by students was often ambiguous or unclear. Writing a single coordinate next to a point does not indicate whether this is a x or y value and this is crucial in the context of this question. Equally for those expressing their reasoning with lengths, it was important to clearly show whether the vertical or horizontal length was being labelled.

QUESTION 5

(a) This question was done very well, with many students making a great attempt at drawing an accurate graph. Partial marks were awarded when the domain used was incorrect, or the period of the function was wrong.

(b) Many students found this item more challenging. The question required the solutions to be labelled on the graph, with the number of solutions also being stated. A considerable number of students did not do both. Allowance was made if the student did not know where $-\sqrt{3}$ was located, but not if the student was trying to locate $\frac{-\sqrt{3}}{2}$ instead. Using $y = \frac{-\sqrt{3}}{2}$ was a common error, as students were trying for an algebraic solution, forgetting that the graph drawn was $y = 2\sin(4x)$ rather than $y = \sin(4x)$ which is what you would obtain after algebraic rearrangement. There was no need to use any algebra at all to complete this item.

QUESTION 6

(a) This item was successfully attempted by many students. The most common error was not realising that the shape of the cos graph was reflected, so that the constant a needed to be negative. There were also other errors with the n value when students merely assumed that n is the same as the period.

(b) Most students were able to work out the relevant equation to solve. Fewer students could solve this well, especially if their incorrect equation from item (a) proved much harder to solve for t than the correct equation for M.

DIFFERENTIAL CALCULUS (SECTION A)

QUESTION 7

(a) This item was well done by students, but marred in some cases with an incorrect cancelling of x.

$$\frac{x\cos(x)-\sin(x)}{x^2} = \frac{x\cos(x)}{x^2} - \frac{\sin(x)}{x^2} = \frac{\cos(x)}{x} - \frac{\sin(x)}{x^2} \quad \text{NOT} \quad \frac{\cos(x)-\sin(x)}{x}.$$

(b) The context of this item was poorly comprehended by most students, although markers were pleased to see that many gave it a valiant attempt rather than leaving it blank.

QUESTION 8

These derivative graphs were quite well done by students.

(a) The polynomial-like graph enjoyed a greater level of success.

(b) Common errors for the derivative graph of the hyperbola-like graph were failing to realise the gradient would always be positive (except at the common vertical asymptote) and not seeing that as $x \to \infty$ then $|f'(x)| \to 0$ meaning there was a horizontal asymptote along the x-axis.

QUESTION 9

This question was reasonably well done by students, provided they successfully differentiated $f(x) = e^x + e$ as $f'(x) = e^x$ (acknowledging e is a constant) and then solved $f'(x) = e$ as the gradient of the tangent (NOT $f'(x) = 0$). Students also needed to find the y coordinate using $f(x)$, since the question asked for a point.

QUESTION 10

Despite its succinctly-worded delivery, this question had a lot to it. Responses had scope for improvement in many cases. For example, getting the correct sign for each of the three terms of the derivative (taking into account $\frac{1}{x}, \frac{1}{x^2}$ have indices -1, -2 respectively) and correctly solving the derivative equation

$$0 = \frac{1}{x} + \frac{1}{x^2} - \frac{2}{x^3}$$

as a partially correct approach of showing $f'(1) = 0$ via substitution does not demonstrate that there are no other stationary points. One approach here was to substitute $a = \frac{1}{x}$ to get $0 = a + a^2 - 2a^3$ which, when implemented correctly, gave a correct result. A more economical approach is to `clear the denominators' (given our discomfort with fractions) by multiplying all terms in the equation by x^3 to obtain $0 = x^2 + x - 2$.

The solution $x = -2$ needed to explicitly be removed from consideration as $\ln(x)$ requires $x > 0$. Many students omitted to show that $f(1) = 0$ for the full coordinate. The classification of the stationary point was generally done well, however, if using a gradient table, the `left' trial value of x should be in the domain of $f(x)$ that is greater than 0.

INTEGRAL CALCULUS (SECTION A)

QUESTION 11

(a) This item was done extremely well by most students. The most common error was assuming that any expression involving x in the denominator of a fraction integrates to $\ln(something)$. This is only true if the expression to be integrated is $\frac{k}{ax+b}$.

(b) This item was also attempted with high success by most students. The most common mistake was trying to integrate the numerator and denominator of the integrand separately, rather than simplifying the fraction first before integrating.

(c) This too was done well, provided students were able to recognise that the $cos^2(x) + sin^2(x)$ expression becomes 1. If students did not make this connection, then the future lines in their working out were often tragically incorrect as integrating $cos^2(x)$ or $sin^2(x)$ successfully is beyond the scope of the course.

QUESTION 12

(a) This item was done very well. The most common error, which detracted slightly from full marks, was leaving off the absolute value signs around the argument of the log expression.

(b) Students achieved mixed success with this item. Items (iii) and (iv) proved the most problematic in terms of correct responses.

QUESTION 13

A fair number of students were able to get to the correct answer. The most common error was not in substituting in the correct expression for $f'(x)$, but instead in trying to incorrectly break up the expression inside the square root into parts. Students who realised they were just integrating $(1+x)^{1/2}$ normally were successful at following through. There was also some confusion in how to deal with $4^{3/2}$, with some students thinking that the simplification was $\sqrt[3]{4^2}$ instead of the correct $\sqrt[2]{4^3}$.

PROBABILITY (SECTION A)

QUESTION 14

(a) This item was generally done well, with most students obtaining full marks. Partial marks were occasionally lost for showing no working. Students are advised to double-check their answer by looking at the mean value they obtained and checking it against the symmetry of the distribution shown.

(b) This item was also well done, although some students found $\Pr(X < 32)$ which was the opposite of the required proportion.

QUESTION 15

(a) While many students realised that this required binomial expansion, results could have been improved by realising that $(1-p)^0 = 1$ and solving $p^3 = \frac{1}{27}$ gives $p = \frac{1}{3}$ as the cube root. Other common errors included missing Pascal coefficients and resolving fractions incorrectly.

(b) Aside from the y-axis scale, this item was well done. Many students who struggled with part a) were able to obtain full marks by representing the distribution with their errors carried forward. Improvements could be made in understanding how to represent a discrete distribution, since continuous curves were often drawn.

(c) Students struggled to interpret the range of probabilities in this item, as many included $\Pr(X = 0)$ and/or $\Pr(X = 3)$ rather than adding the two required.

QUESTION 16

(a) This item was poorly answered. Students could have improved their result by first identifying the sample space of 15 outcomes and then working out just how many of these 15 were winning outcomes. Those who were able to establish this were then able to find $E(X)$.

(b) Students were often able to obtain marks when their errors were carried forward here. The most common mistake seen was multiplying rather than dividing ($15 \div \frac{1}{5} = 75$ was the correct approach).

FUNCTION STUDY (SECTION B)

QUESTION 17

While many students were very successful with this, there were a number of attempts where students reflected about the vertical asymptote of $f(x)$ rather than around the y axis. This was a significant error resulting in considerable (but not total) loss of marks. It was important to see the asymptotes of $f(-x)$ labelled correctly as well as the x-intercept being labelled as $x = -p$ rather than $x = p$. Another common mistake was in thinking $f(-x)$ is the same as the inverse function $f^{-1}(x)$, so that students incorrectly reflected the graph about the line $y = x$.

QUESTION 18

The vast majority of students were able to get to the line of reasoning $(x-2)(x+5) = 8$, correctly applying the relevant log law. When listing the solutions as $x = -6$ or $x = 3$, students rejected $x = -6$ merely by stating that you "cannot have negative logs". This attracted a small marking penalty as this was insufficient reasoning. Students needed to show that at least one of the log expressions $\log_3(x-2)$ or $\log_3(x+5)$ became undefined when $x = -6$.

QUESTION 19

Most students were able to correctly write an expression for $f(f^{-1}(x))$ and $f^{-1}(f(x))$. However, simplifying these to just x proved more challenging for many students. If students wrote an expression (often derived from a calculator) that included absolute value expressions, then they needed to give the domain for which this would simplify to x.

QUESTION 20

(a) Students who determined the turning points of the function within the given domain were highly successful at correctly listing out the relevant maximal domain. It was pleasing that most students realised that endpoints were included in the domain, even if the endpoint values were incorrect.

(b) Students who completed item (a) successfully were often able to do item (b) well too. However, full marks were not awarded unless there was some explanation that the range of the function $f(x)$ is the same as the domain of the inverse function $f^{-1}(x)$.

QUESTION 21

(a) This item was completed very successfully by most students. They correctly identified the relevant asymptotes from the graph and computed the road width at large distances from the origin.

(b) This too was done well, with the most common errors involving the determination of a. This was due to arithmetic mistakes or by forgetting the 'square' in the 'squared' part of the expression.

(c) Many students correctly identified that North Road ends at $y = 30 + 64 = 94$. The most common error was in assuming that North Road was being modelled by $g(x)$ instead of $f(x)$.

CIRCULAR FUNCTIONS (SECTION B)

QUESTION 22

Students did this question largely well, either using the $cos^2(x) + sin^2(x)$ formula or by drawing the relevant triangle. The most common mistake was in forgetting that θ is in the second quadrant of the unit circle, so the cos value needs to be negative.

QUESTION 23

(a) This item was very challenging for students, with very few students gaining full marks. It was expected that some explanation was required to show that $\sin(-\pi + \theta) = -\sin(\theta)$ and that $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin(\theta)$ either through words or via a diagram. While many students partially completed the question, only a small minority provided the relevant explanation.

(b) This item was done moderately well. Some students assumed that the question was asking to convert $\frac{1}{4}$ into radians, instead of evaluating $sin^{-1}\left(\frac{1}{4}\right)$.

QUESTION 24

This question proved a success with many students. Marks were awarded for errors carried forward when students did not determine that the vertical asymptote was at $x = \frac{3}{8}$ (with $x = \frac{3}{4}$ being the most common mistake). Some students forgot to include y coordinates or write out the final point values.

QUESTION 25

(a) This item was done very well by most students, except for the horizontal translation amount which was often incorrect. Another common error was in mixing up the midline and amplitude values so that the values of a and c were swapped. Errors in the period also led to n being incorrect.

(b) These multiple-choice answers proved challenging for some students. 'One' and 'two' were accepted as possible answers for n. The correct answer is 'two' since both $n = 2$ and $n = -2$ are possible values.

DIFFERENTIAL CALCULUS (SECTION B)

QUESTION 26

(a) This item was well done. Students usually realised that `show that' required some detail in determining $P'(x)$ from $P(x)$.

(b) Successful students used the sketch of $P'(x)$ (as instructed) to argue that in the neighbourhood of $x = 25$, (moving from left to right) $P'(x)$ goes $+, 0, -$. Most students opted for an algebraic approach on the equation of $P'(x)$ and gained only partial marks.

QUESTION 27

Many students interpreted this question to mean 'show that y is decreasing'. Students who were successful in this received most marks if they obtained $\frac{dy}{dx} = -10$ when $x = \frac{\pi}{6}$ and explained that a negative gradient corresponds to a decreasing function.

The product and chain rule were usually well done. Few students recognised that showing the gradient was decreasing involved finding the second derivative. Substituting $\frac{\pi}{6}$ into $\frac{d^2y}{dx^2}$ gave a negative answer, meaning the gradient, $\frac{dy}{dx}$, was decreasing.

QUESTION 28

(a) This was a classic style of question and students, being spared the overheads of showing the correctness of the time equation $T(x)$, were able to use their calculator to navigate it.

(b) This was well done. Units were needed in item (b) and almost uniformly given. A few students truncated 1.06666666 to 1.06 (incorrect) rather than rounding to 1.07 (if they chose to quote to 2dp).

QUESTION 29

(a) This item was well done by students who knew that a normal line is perpendicular to a curve. More care might have been taken to demonstrate the perpendicular using a right angle symbol or drawing the line so that the angle is close to a right angle.

(b) Many students understood what was required in this item, although often a by-hand approach was chosen in finding $f'(\frac{\sqrt{\ln(2)}}{\sqrt{2}})$ which proved perilous. Markers required some acknowledgement of how the normal gradient could be obtained from the tangent to award full marks.

(c) Successful students used $(x, y) = (\frac{\sqrt{\ln(2)}}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $m = \frac{1}{\sqrt{\ln(2)}}$ to find the equation of the normal. They then verified that $(0,0)$ was on this line which required a little more than `$c = 0'$ (for example `$c = 0'$ means the y −intercept is 0). A lot of students conflated these two steps using $(0,0)$ to find the equation of the normal and then (tautologically) showing that $(0,0)$ lay on the line.

INTEGRAL CALCULUS (SECTION B)

QUESTION 30

This question was done very well. The most common errors were when students did not integrate correctly or when they added rather than subtracted the expressions after substitution.

QUESTION 31

It was clear during the marking process that students interpreted the wording of the question in different ways, so full marks were awarded for students either correctly stating expressions for A and B separately, or instead for the combined area A+B which is a much easier expression.

There were many possible ways the question could have been answered depending on how the areas were broken up. Students needed to take care in the signs before the integral, the terminals used, and the order of subtraction if they were using the $f(x) - g(x)$ or $g(x) - f(x)$ approach in the integrands. If subtracting from the area of the rectangle, then the full rectangular area could be expressed as either $cf(c)$ or $cg(0)$.

QUESTION 32

A fair number of students found this question challenging. When writing $f(x)$ for $g(x)$, the factor 4 was used rather than ¼. Also, the splitting of the integral into two components was often done incorrectly. It is not required (and technically incorrect) to write $units^2$ in your answer as this is not an 'area' question, but marks were not deducted for this.

QUESTION 33

Students who successfully found the intersection point between $f(x)$ and $g(x)$ at $x = \frac{\pi}{3}$ usually had no great trouble with the question. As this is a 'show that' style of question, there was an expectation that students would indicate the integrals being used, the integration step ('square bracket' line).

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QUESTION 34

(a) This item was done very well by most students, with any mistakes being related mainly to incorrect integration or forgetting that the integration constant is $\frac{-1}{10000}$ instead of $\frac{1}{10000}$.

(b) This item was done with an even higher success rate than item (a). Fortuitously, the value of the constant being so small meant that even an incorrect value for it did not reflect in the value of x obtained after rounding to the nearest km.

PROBABILITY (SECTION B)

This section was extremely well done by students who left sufficient time for it. A good area to focus on for future students would be on how to support answers with the appropriate amount of working – not using calculator syntax – which is required to earn full marks.

QUESTION 35

(a) This item was well done with most students recognising that $\sum \Pr(X = x) = 1$ and then using their calculator to solve for a. Students needed to explain why $a = -2$ is not an acceptable solution. A good way to do this was with an example: if $a = -2$ then $\Pr(X = 3) = -2$ (invalid as $0 \leq \Pr(X = x) \leq 1$). Partial marks were awarded to students who substituted $a = \frac{1}{4}$ to show the probabilities added up correctly, but this is not the way to approach 'show that' questions.

(b) Most students used the appropriate formula for variance, however, there were several difficulties in resolving fractional answers or not squaring the appropriate parts. A useful strategy seemed to be calculating the probabilities with $a = \frac{1}{4}$ substituted and writing these values in the table.

QUESTION 36

(a) Students did very well here, although the z axis was often not acknowledged and subsequently this meant that full marks were not often obtained.

(b) This item was also very well done. Students are reminded to include units for standard deviation, and to make sure they show enough working to obtain full marks. Often $z = -2.0537$ was quoted, with no supporting justification on how it was obtained. Calculator syntax should be avoided here where possible.

(c) Students were able to use the standard deviation given to them in the previous item to achieve the correct response by solving on their calculator.

QUESTION 37

(a) This item was very well done.

(b) Students generally had no trouble finding the margin of error by using the formula on the Information Sheet. While latitude was given for rounded answers, markers were once again looking for some evidence of working. The confidence value also needed to be supported with working out, and many students found $\Pr(-\infty < Z < 2.576)$ rather than finding $\Pr(-2.576 < Z < 2.576)$.

(c) Students who got this far were able to establish the correct confidence interval but often missed interpreting it in the context of the tennis scenario, or forgot to include 90% in their description.

Solutions provided on the following page

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Attach your candidate label here

External Assessment 2022

MATHEMATICS METHODS

MTM415117

Section A

Pages	24
Questions	16
Information Sheet	1

Preparation time for this exam: 15 minutes

Suggested working time: 80 minutes

Instructions:

Calculators are not allowed to be used in this section.

Section A will be collected after 80 minutes.

- There are **five (5)** parts to this section.
- Answer **all** questions and **all** items within each question.
- Write your answers in the spaces provided in this exam paper.
  - Spare diagrams have been provided at the end of each part. Indicate in the box provided if you have used the spare diagrams.
- The exam is **three (3)** hours in length. It is suggested that you spend **approximately 80 minutes** in total answering the questions in this section.
- During the first 80 minutes you may move onto Section B, but you cannot use your calculator until told by your supervisor(s).
- The **Mathematics Methods Information Sheet** can be used throughout the exam.
- All answers must be written in **English**.
- You **must** make sure your answers address:
  - Criterion 4 understand polynomial, hyperbolic, exponential and logarithmic functions
  - Criterion 5 understand circular functions
  - Criterion 6 use differential calculus in the study of functions
  - Criterion 7 use integral calculus in the study of functions
  - Criterion 8 understand binomial and normal probability distributions and statistical inference.

Marker Use	
C4	/ 16
C5	/ 16
C6	/ 16
C7	/ 16
C8	/ 16

Part 1

- Attempt **all** questions in this part.
- This part assesses **Criterion 4**.

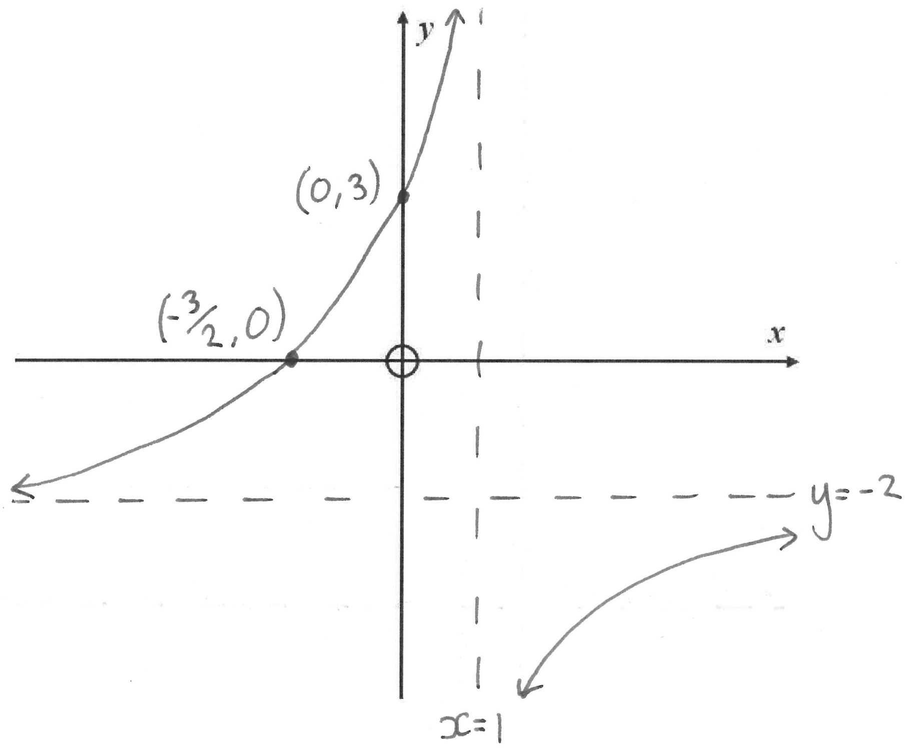
Question 1

Marker use

a) Sketch a graph of $f(x) = \dfrac{-5}{x-1} - 2$ on the axes below.

/3

Label any asymptotes and axes intercepts.



Spare diagram used (✓)

Hyperbola with VA at $x=1$, HA at $y=-2$

y int : $f(0) = \frac{-5}{0-1} - 2$ x int : $0 = \frac{-5}{x-1} - 2$

$= 5 - 2$ $2(x-1) = -5$

$= 3 \quad \therefore (0,3)$ $x = -3/2 \quad \therefore (-3/2, 0)$

b) Hence, determine the domain where $f(x)$ is positive.

/2

$x \in (-3/2, 1)$

Part 1 continues

Part 1 continued

Question 2

A quadratic function, $f(x)$, has a turning point at $(4,1)$ and intersects the y axis at 2.

a) Determine the equation for this parabolic function in the form $f(x) = a(x-h)^2 + k$.

TP gives $(h,k) = (4,1)$

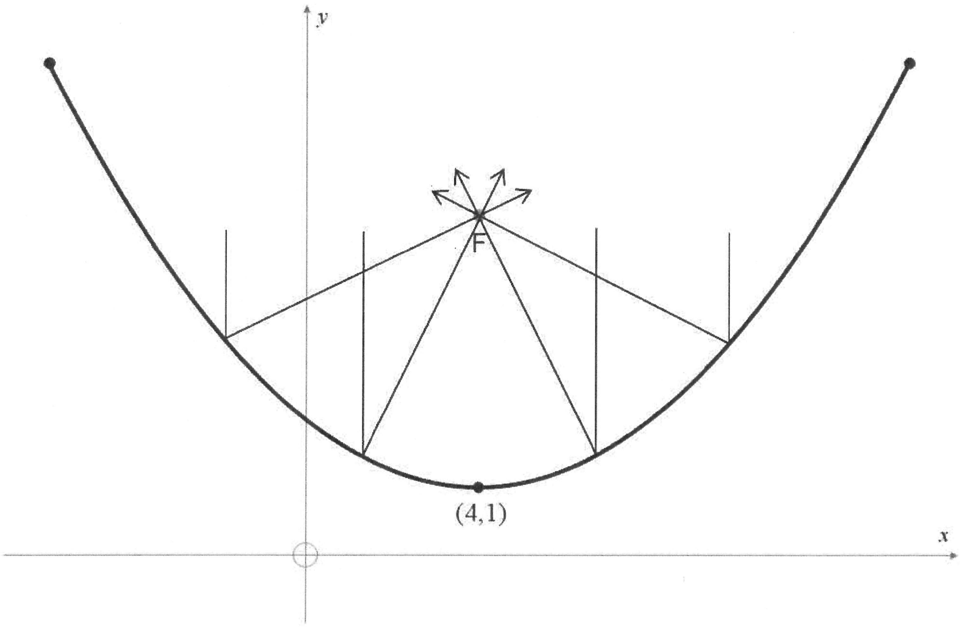
Sub $(0,2)$: $2 = a(0-4)^2 + 1$

$1 = 16a$

$\therefore a = \frac{1}{16}$ $\qquad \therefore y = \frac{1}{16}(x-4)^2 + 1$

Parabolic solar devices reflect direct sunlight to a focal point.

The diagram below illustrates this focal point F, for $f(x)$ where $x \in [-6,14]$.



b) Determine the range of $f(x)$ for the sketch above.

$f(14) = f(-6) = \frac{1}{16}(-6-4)^2 + 1$

$= \frac{100}{16} + 1$

$= \frac{116}{16}$ or $\frac{29}{4}$ $\qquad \therefore y \in [1, \frac{29}{4}]$

c) Determine the coordinate for the focal point, F which is given by $\left(h, k + \frac{1}{4a}\right)$.

$F = \left(4, 1 + \frac{1}{4(\frac{1}{16})}\right)$

$= (4, 1 + \frac{16}{4}) = (4, 5)$

Marker use

2

2

1

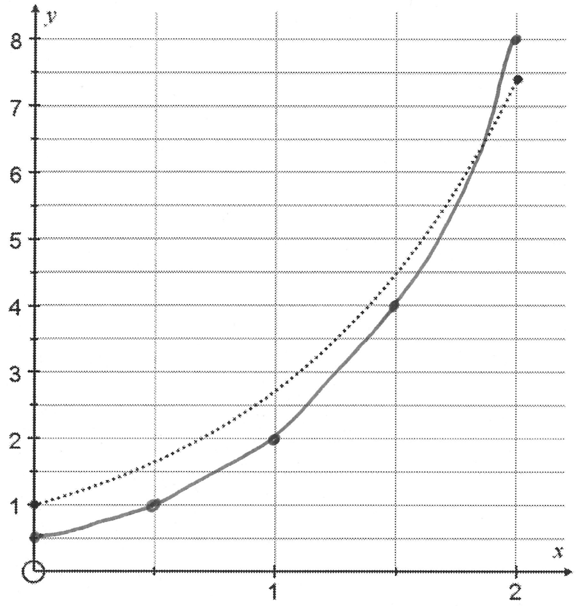
Part 1 continues

Part 1 continued

Marker use

Question 3

A graph of $y = e^x$ for $x \in [0, 2]$ is shown below.



Spare diagram used (✓)

a) On the same axes, **plot** points to accurately graph $y = 2^{(2x-1)}$ over the same domain.
No working required.

At $x=0$, $y=2^{-1} = \frac{1}{2}$ | $x=\frac{1}{2}$, $y=2^{1-1} = 1$ | $x=1$, $y=2^{2-1} = 2$ | $x=\frac{3}{2}$, $y=2^{3-1} = 4$ | $x=2$, $y=2^{4-1} = 8$

2

b) Hence, determine an **approximate** graphical solution for $e^x = 2^{(2x-1)}$.

$x \approx 1.8$

1

c) Determine an **exact** solution for $e^x = 2^{(2x-1)}$ in terms of natural logarithms.

$\ln(e^x) = \ln(2^{2x-1})$

$x \ln(e) = (2x-1)\ln(2)$

$x \times 1 = 2x\ln(2) - \ln(2)$

$x - 2x\ln(2) = -\ln(2)$

$x(1-2\ln(2)) = -\ln(2)$

$\therefore \quad x = \dfrac{-\ln(2)}{1-2\ln(2)}$ or $\dfrac{\ln(2)}{2\ln(2)-1}$ or $\dfrac{\ln(2)}{\ln(4)-1}$

3

Total C4

16

Part 2

- Attempt **all** questions in this part.
- This part assesses **Criterion 5**.

Marker use

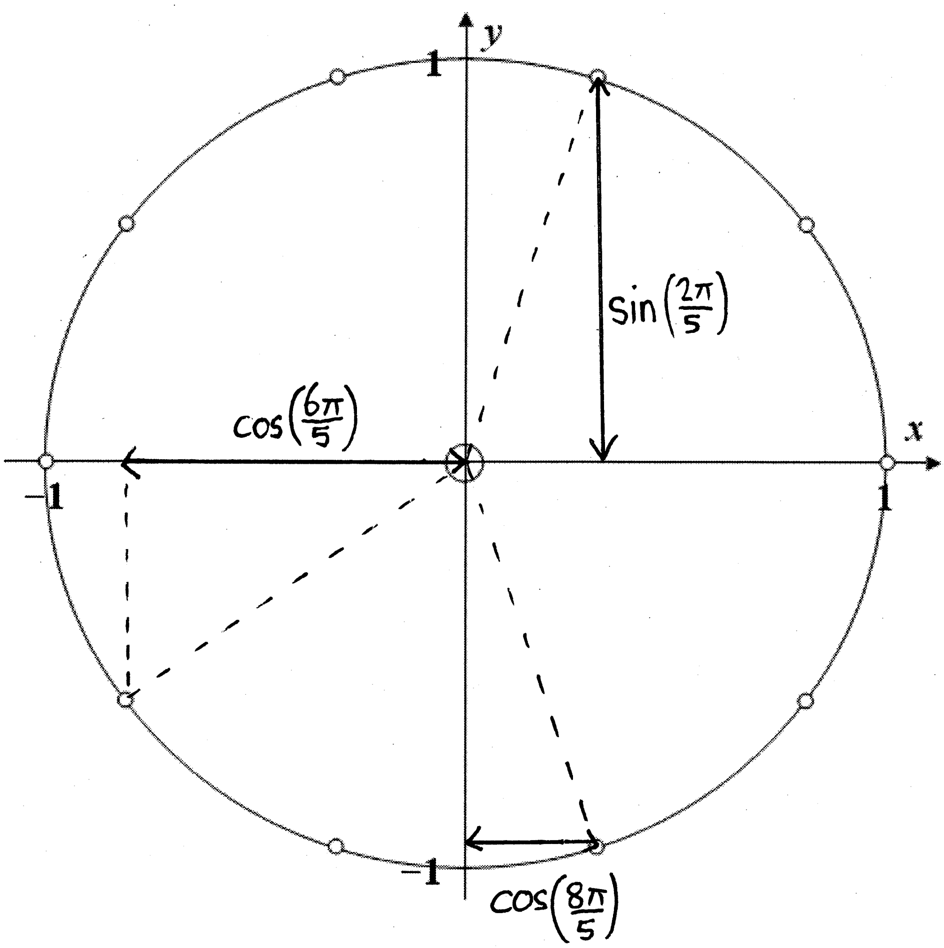
Question 4

a) Convert $\frac{8\pi}{5}$ radians into degrees. 1

$\frac{8\pi}{5} \times \frac{180}{\pi} = 8 \times 36 = 288^\circ$

b) Accurately label $\sin\left(\frac{2\pi}{5}\right)$, $\cos\left(\frac{6\pi}{5}\right)$ and $\cos\left(\frac{8\pi}{5}\right)$ on the unit circle diagram below. 3

Dots are evenly spaced around the unit circle.



Spare diagram used (✓)

☐

Part 2 continues

Part 2 continued

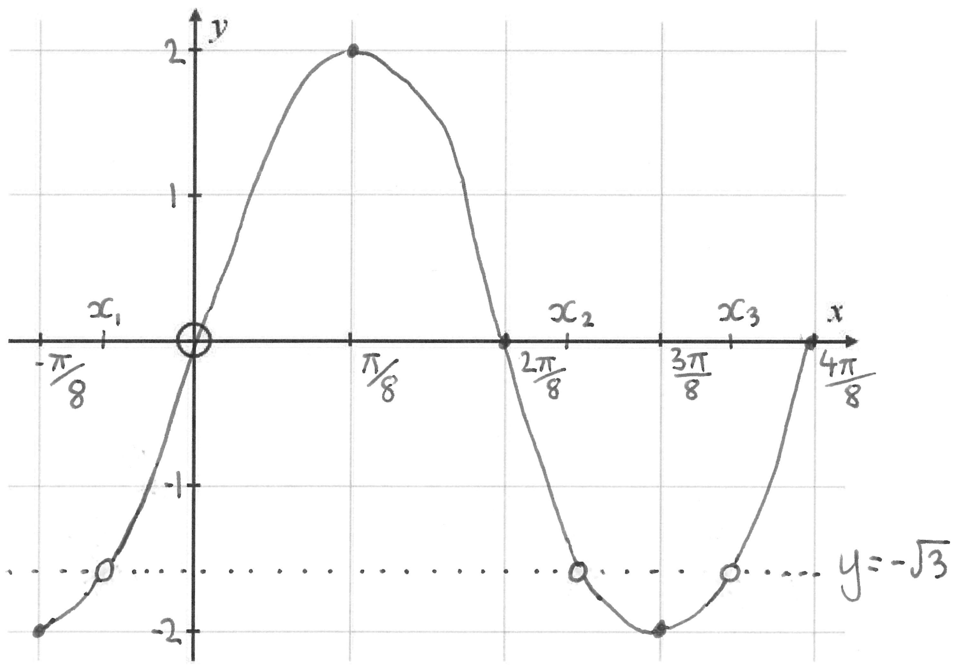
Question 5

a) Label the axes below with appropriate scales and accurately sketch the graph of $y = 2\sin(4x)$ over the domain $x \in \left[\frac{-\pi}{8}, \frac{\pi}{2}\right]$.

Period $= \frac{2\pi}{4} = \frac{4\pi}{8}$

Amplitude $= 2$

Reflected



Spare diagram used (✓)

b) Hence, use your graph to determine the **number** of solutions to the equation $2\sin(4x) = -\sqrt{3}$ for the domain $x \in \left[\frac{-\pi}{8}, \frac{\pi}{2}\right]$.

Label these solutions on the sketch.

There is no need to determine values for these solutions.

3 solutions, labelled x_1, x_2 and x_3 above.

Marker use

/4

/2

Part 2 continues

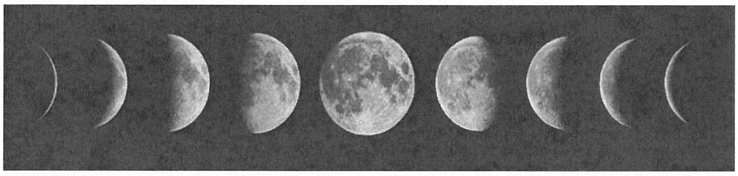
Part 2 continued

Marker use

Question 6

The visibility of the moon from earth varies from 0% to 100% and back to 0% over 30 days.

The images and graph below model this variation.



a) Determine a function of the form $M = a\cos(bt) + c$ where M represents the percentage of the moon visible, t days after a new moon. /3

$|a| = \frac{Max - Min}{2} = \frac{100-0}{2} = 50 \quad \therefore a = -50$ (Reflected)

$c = \frac{Max + Min}{2} = \frac{100+0}{2} = 50$

$Period = 30 = \frac{2\pi}{b} \quad \therefore M = -50\cos\left(\frac{\pi}{15}t\right) + 50$

$\therefore b = \frac{\pi}{15}$

Question 6 continues

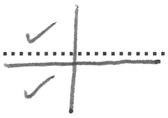
Question 6 continued

b) Hence, determine when more than 75% of the moon is visible.

$75 = -50\cos\left(\frac{\pi}{15}t\right) + 50$

$25 = -50\cos\left(\frac{\pi}{15}t\right)$

$-\frac{1}{2} = \cos\left(\frac{\pi}{15}t\right)$



Base Angle $= \pi/3$

$\therefore \frac{\pi}{15}t = \pi - \pi/3,\ \pi + \pi/3$

$= 2\pi/3,\ 4\pi/3$

$\pi t = 10\pi,\ 20\pi$

$t = 10, 20$

$\therefore t \in (10, 20)$

OR between day 10 and day 20

Marker use

/3

Total C5

/16

Part 3

- Attempt **all** questions in this part.
- This part assesses **Criterion 6**.

Marker use

Question 7

a) Given $f(x)=\dfrac{\sin(x)}{x}$, use the quotient rule to show that $f'(\pi)=-\dfrac{1}{\pi}$. /2

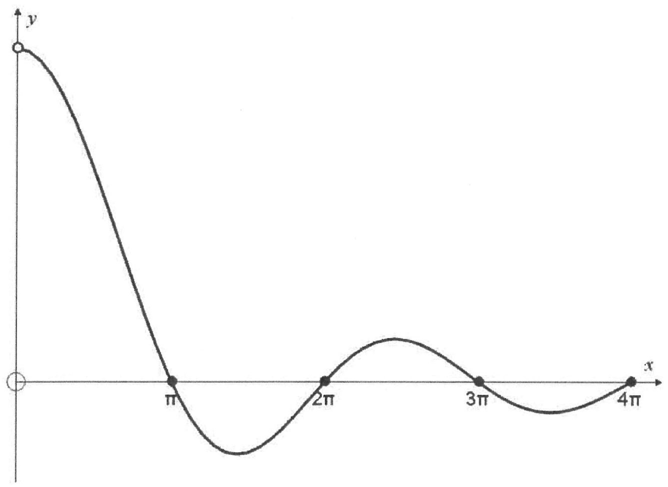
$f'(x) = \dfrac{x \cdot \cos(x) - 1 \times \sin(x)}{x^2}$

$f'(\pi) = \dfrac{\pi \cos(\pi) - \sin(\pi)}{\pi^2}$

$= \dfrac{-\pi - 0}{\pi^2}$

$= -1/\pi$

Consider the following graph and derivative values for $f(x)=\dfrac{\sin(x)}{x}$.



x	π	2π	3π	4π
$f'(x)$	$-\dfrac{1}{\pi}$	$\dfrac{1}{2\pi}$	$-\dfrac{1}{3\pi}$	$\dfrac{1}{4\pi}$

b) State a **variation**, across the tabled $f'(x)$ values, relevant to the graph having:

i. Alternating minima and maxima. /1

$f'(x)$ oscillates -ve / +ve / -ve / +ve

ii. A decreasing amplitude. /1

$|f'(x)|$ is decreasing

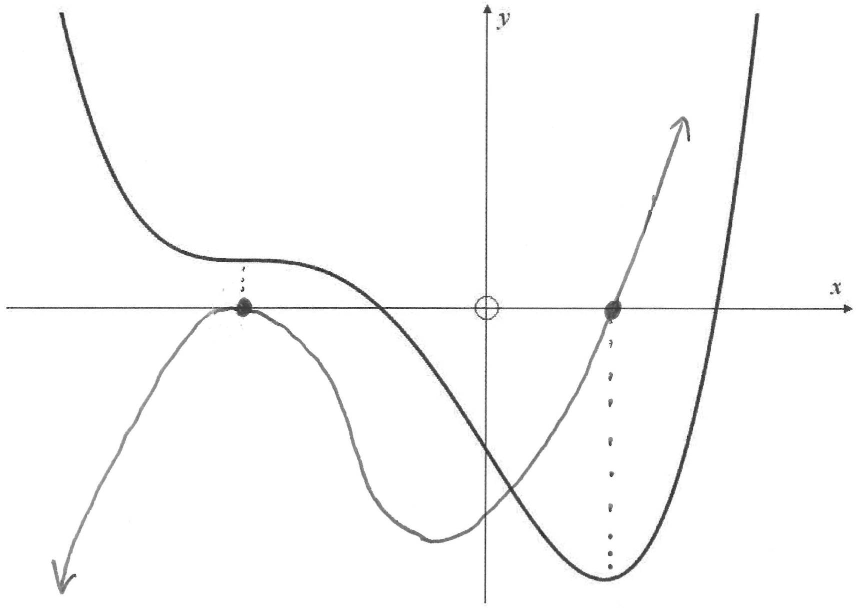
Part 3 continues

Part 3 continued

Question 8

Sketch possible **derivative graphs** on the respective axes for the functions below.

a)

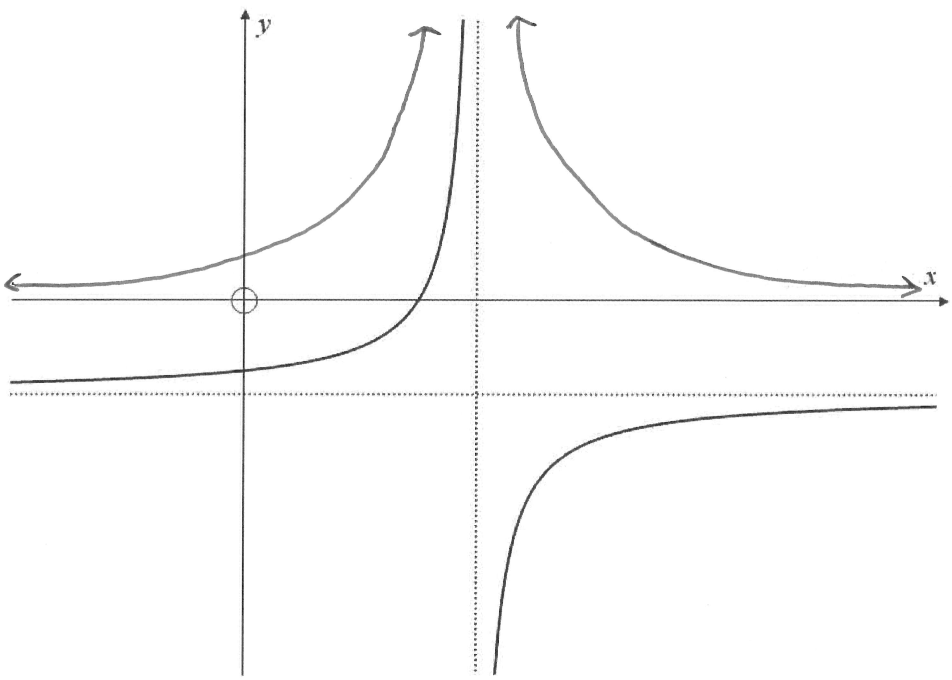


Spare diagram used (✓)

/ 2

b)

/ 2



Spare diagram used (✓)

Marker use

Part 3 continues

Part 3 continued

Marker use

Question 9

Determine the point where the line $ex - y + e = 0$ is a tangent to $f(x) = e^x + e$. /3

$y = ex + x$ gives $m_T = e$

$f'(x) = e^x + 0$

Let $f'(x) = e$

$\therefore\ e = e^x$

$\therefore\ x = 1$

$f(1) = e^1 + e = 2e$ $\therefore\ (1, 2e)$

Question 10

Show that $f(x) = \ln(x) - \frac{1}{x} + \frac{1}{x^2}$ has only **one** stationary point at $(1,0)$. Justify its nature. /5

$f'(x) = \frac{1}{x} - \frac{-1}{x^2} + \frac{-2}{x^3}$

SP's: $0 = 1/x + \frac{1}{x^2} - 2/x^3$ $(\times x^3 \text{ as } x \neq 0)$

$= x^2 + x - 2$

$= (x+2)(x-1)$

$\therefore\ x = -2$ or $x = 1$

Invalid as $\ln(-2)$

$f(1) = \ln(1) - \frac{1}{1} + \frac{1}{1}$

$= 0$

$f''(x) = -1/x^2 - 2/x^3 + 6/x^4$

$f''(1) = -1 - 2 + 6$

$= 3$

> 0

x	$1/2$	1	2
$f'(x)$	-10	0	$1/2$
Shape	╲	—	╱

$\therefore\ (1,0)$ is a local minimum (and absolute)

Total C6 /16

Part 4

- Attempt **all** questions in this part.
- This part assesses **Criterion 7**.

Question 11

Marker use

Apply the appropriate index law(s) or basic identity before determining the following integrals.

a) $\int_1^9 \frac{1}{\sqrt{x}}\,dx$.

$= \int_1^9 x^{-1/2}\,dx$

$= \left[\frac{x^{1/2}}{1/2}\right]_1^9$

$= 2\left[\sqrt{x}\right]_1^9$

$= 2(\sqrt{9} - \sqrt{1})$

$= 2(3-1)$

$= 4$

/ 2

b) $\int \frac{e^{5x}}{(e^x)^2}\,dx$.

$= \int \frac{e^{5x}}{e^{2x}}\,dx$

$= \int e^{5x-2x}\,dx$

$= \int e^{3x}\,dx$

$= \frac{1}{3}e^{3x} + c$

/ 2

c) $\int_2^5 (3\cos^2 x + 3\sin^2 x)\,dx$.

$= 3\int_2^5 \cos^2 x + \sin^2 x\,dx$

$= 3\int_2^5 1\,dx$

$= 3\left[x\right]_2^5$

$= 3(5-2)$

$= 9$

/ 2

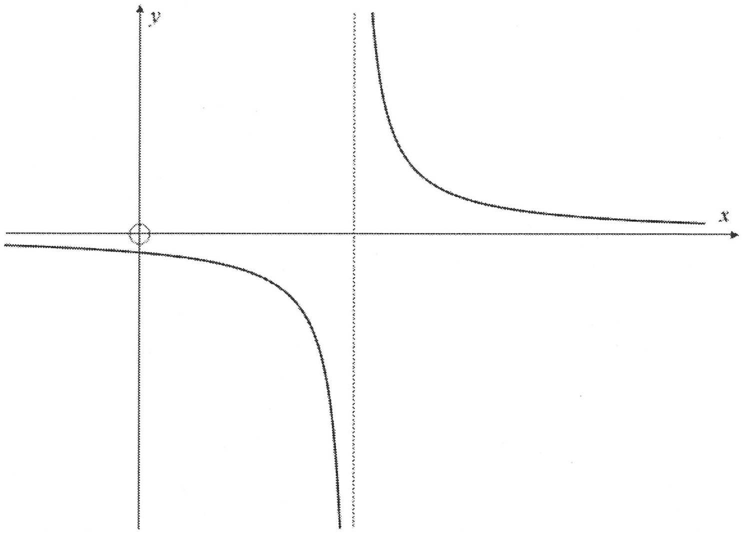
Part 4 continues

Marker use

Question 12

The function $f(x) = \dfrac{1}{x-2}$

is sketched to the right.



a) State $\int \dfrac{1}{x-2}\,dx$. 2

$= \ln|x-2| + c$

b) Circle the appropriate word to the right of the following definite intergrals to identify whether they are: *POSITIVE*, *NEGATIVE*, *ZERO* or *UNDEFINED*. 4

No working required.

i. $\int_3^5 f(x)\,dx$ *POSITIVE* *NEGATIVE* *ZERO* *UNDEFINED*

ii. $\int_{-2}^0 f(x)\,dx$ *POSITIVE* *NEGATIVE* *ZERO* *UNDEFINED*

iii. $\int_1^3 f(x)\,dx$ *POSITIVE* *NEGATIVE* *ZERO* *UNDEFINED*

iv. $\int_0^1 f(x)\,dx + \int_3^4 f(x)\,dx$ *POSITIVE* *NEGATIVE* *ZERO* *UNDEFINED*

Part 4 continues

Part 4 continued

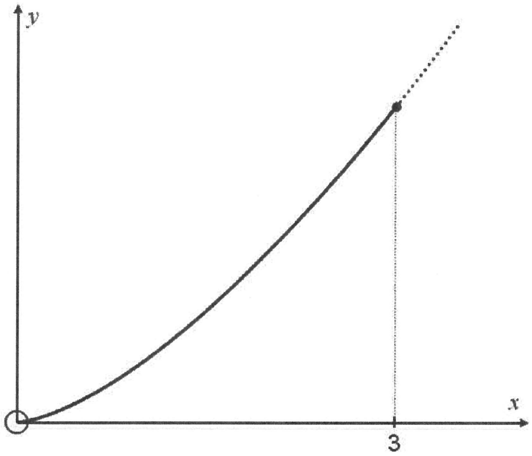
Marker use

Question 13

For a continuous function $f(x)$, the **length** of the curved section over $x \in [a, b]$, is given by:

$$\int_a^b \sqrt{1+[f'(x)]^2}\,dx.$$

The function $f(x)=\frac{2\sqrt{x^3}}{3}$ is graphed below with the curved section over $x \in [0,3]$ highlighted.



The derivative of $f(x)=\frac{2\sqrt{x^3}}{3}$ equals $\sqrt{x}$.

Determine the **length** of the curved section highlighted above.

$\text{Length} = \int_0^3 \sqrt{1+(\sqrt{x})^2}\,dx$ as $a=0$, $b=3$, $f'(x)=\sqrt{x}$

$= \int_0^3 \sqrt{1+x}\,dx$

$= \int_0^3 (1+x)^{1/2}\,dx$

$= \left[\frac{(1+x)^{3/2}}{3/2}\right]_0^3$

$= \frac{2}{3}\left((1+3)^{3/2} - (1)^{3/2}\right)$

$= \frac{2}{3}\left(\sqrt{4}^3 - 1\right)$

$= \frac{2}{3}(8-1)$

$= \frac{14}{3}$ units

/4

Total C7

/16

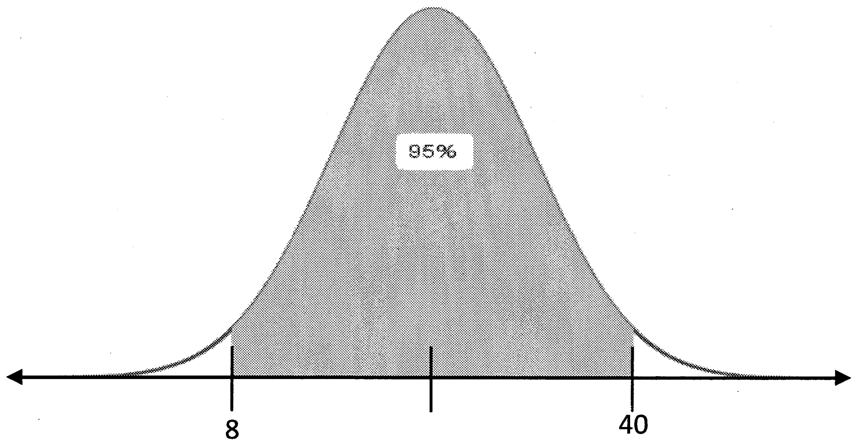
Part 5

- Attempt **all questions** in this part.
- This part assesses **Criterion 8**.

Question 14

Marker use

The shaded area is **approximately** 95% of the normal distribution below and symmetrical about the mean.



a) Determine approximations for the mean and standard deviation for this distribution. /2

$\mu = \frac{40+8}{2}$ 95% means $40 = \mu + 2\sigma$

$2\sigma = 40 - 24$

$= 24$ $= 16$

$\therefore \sigma = 8$

b) For this distribution, determine an approximation for $\Pr(X > 32)$. /2

$\Pr(X > 32) = \Pr(X > \mu + \sigma)$

$= \frac{100 - 68}{2}$

$= 16\%$ or 0.16

Part 5 continues

Part 5 continued

Marker use

Question 15

A partially completed table for a binomial distribution $X \sim Bi(3, p)$ is given below:

x	0	1	2	3
$\Pr(X = x)$	$\frac{8}{27}$	$\frac{12}{27}$ $\left(\frac{4}{9}\right)$	$\frac{6}{27}$ $\left(\frac{2}{9}\right)$	$\frac{1}{27}$

a) Determine p for this distribution and hence, complete all entries in the table. /4

$\frac{1}{27} = {}^3C_3\,(p)^3\,(1-p)^0$

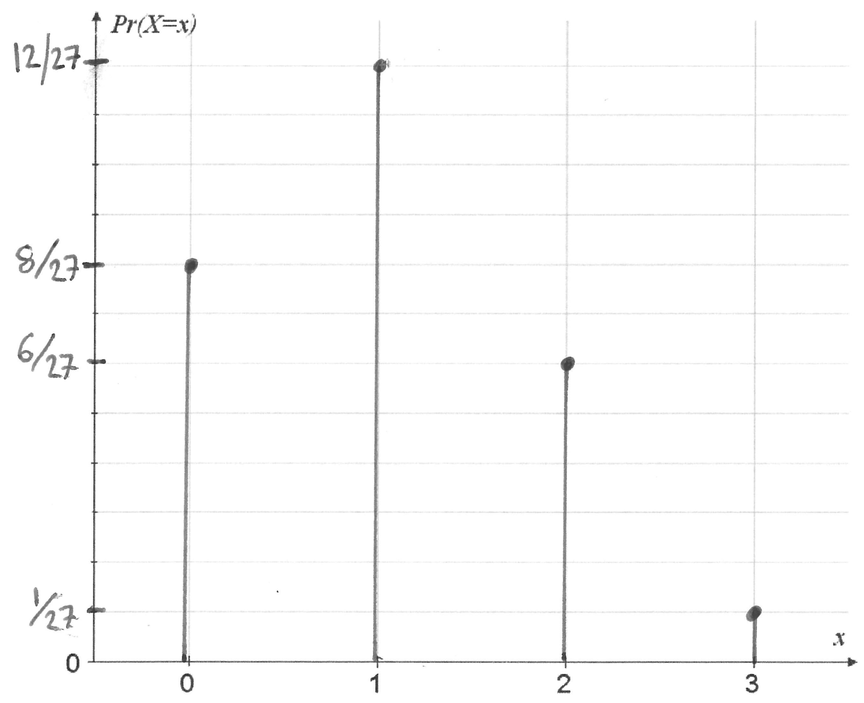
$\frac{1}{27} = p^3$

$\therefore p = \frac{1}{3}$ $\quad \Pr(X=0) = {}^3C_0 \left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^3 = \frac{8}{27}$

$\Pr(X=1) = {}^3C_1 \left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2 = 3 \cdot \frac{1}{3} \cdot \frac{4}{9} = \frac{12}{27}$

$\Pr(X=2) = {}^3C_2 \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^1 = 3 \cdot \frac{1}{9} \cdot \frac{2}{3} = \frac{6}{27}$

b) Choose an appropriate **fractional** scale for the y axis. Represent this distribution on the diagram below. /2



Spare diagram used (✓) ☐

c) Determine $\Pr(0 < X < 3)$ for this distribution. /1

$= \Pr(X=1) + \Pr(X=2)$

$= \frac{12}{27} + \frac{6}{27} = \frac{18}{27}$ OR $\frac{2}{3}$

Part 5 continues

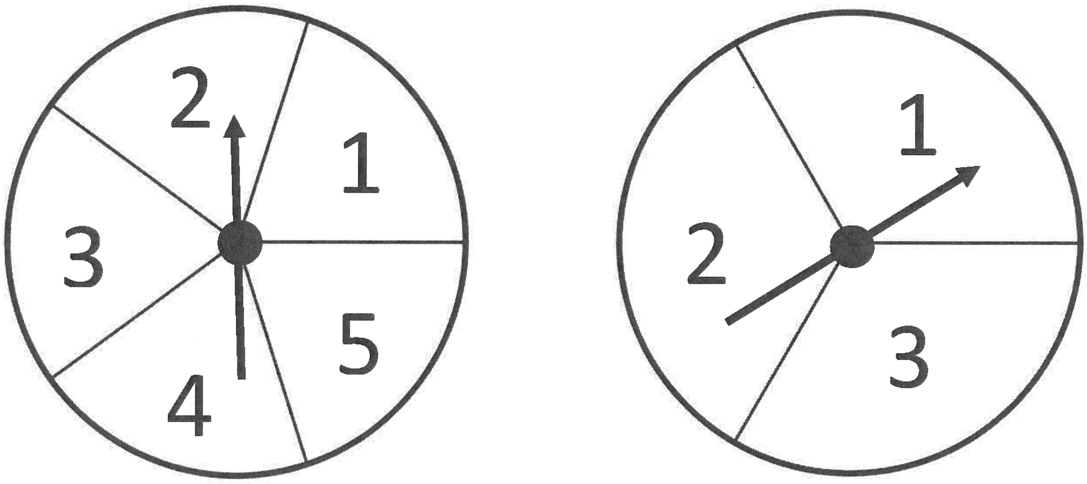
Part 5 continued

Marker use

Question 16

The diagrams below represent two different "**spinners**" to be used in a game of chance.

All sectors within each spinner are the same size.



In a game, a player spins the arrows which point to a numbered sector when stopped.

If the numbers are **different**, the player **loses** 1 token.

If the numbers are the **same**, the player **gains** 3 tokens.

a) Calculate the expected value to show this game results in a net loss of tokens.

There are 15 outcomes that are equally likely.

(1,1), (2,2) and (3,3) win, so $Pr(WIN) = 3/15$

$= 1/5$

$Pr(LOSE) = 1 - 1/5$

$= 4/5$

$E(X) = 1/5(+3) + 4/5(-1)$

$= 3/5 - 4/5$

$= -1/5$ (Net loss of a fifth of a token is expected).

/4

b) A player starts with 15 tokens. Calculate the likely number of games before all tokens are lost.

$-15 = -1/5 \times$ (Number of games)

∴ 75 games required to lose all 15 tokens.

/1

Total C8

/16

OFFICE OF TASMANIAN ASSESSMENT, STANDARDS & CERTIFICATION

External Assessment 2022

Attach your candidate label here

MATHEMATICS METHODS

MTM415117

Section B

Pages	28
Questions	21
Information Sheet	1

Suggested working time: 100 minutes

Instructions:

Calculators are allowed to be used in this section.

- There are **five (5)** parts to this section.
- Answer **all** questions and **all** items within each question.
- Write your answers in the spaces provided in this exam paper.
  - Spare diagrams have been provided at the end of each part. Indicate in the box provided if you have used the spare diagrams.
- The exam is **three (3)** hours in length. It is suggested that you spend **approximately 100 minutes** in total answering the questions in this section.
- During the first 80 minutes you may move onto Section B, but you cannot use your calculator until told by your supervisor(s).
- The **Mathematics Methods Information Sheet** can be used throughout the exam.
- All answers must be written in **English**.
- You **must** make sure your answers address:
  - Criterion 4 understand polynomial, hyperbolic, exponential and logarithmic functions
  - Criterion 5 understand circular functions
  - Criterion 6 use differential calculus in the study of functions
  - Criterion 7 use integral calculus in the study of functions
  - Criterion 8 understand binomial and normal probability distributions and statistical inference.

Marker Use	
C4	/ 20
C5	/ 20
C6	/ 20
C7	/ 20
C8	/ 20

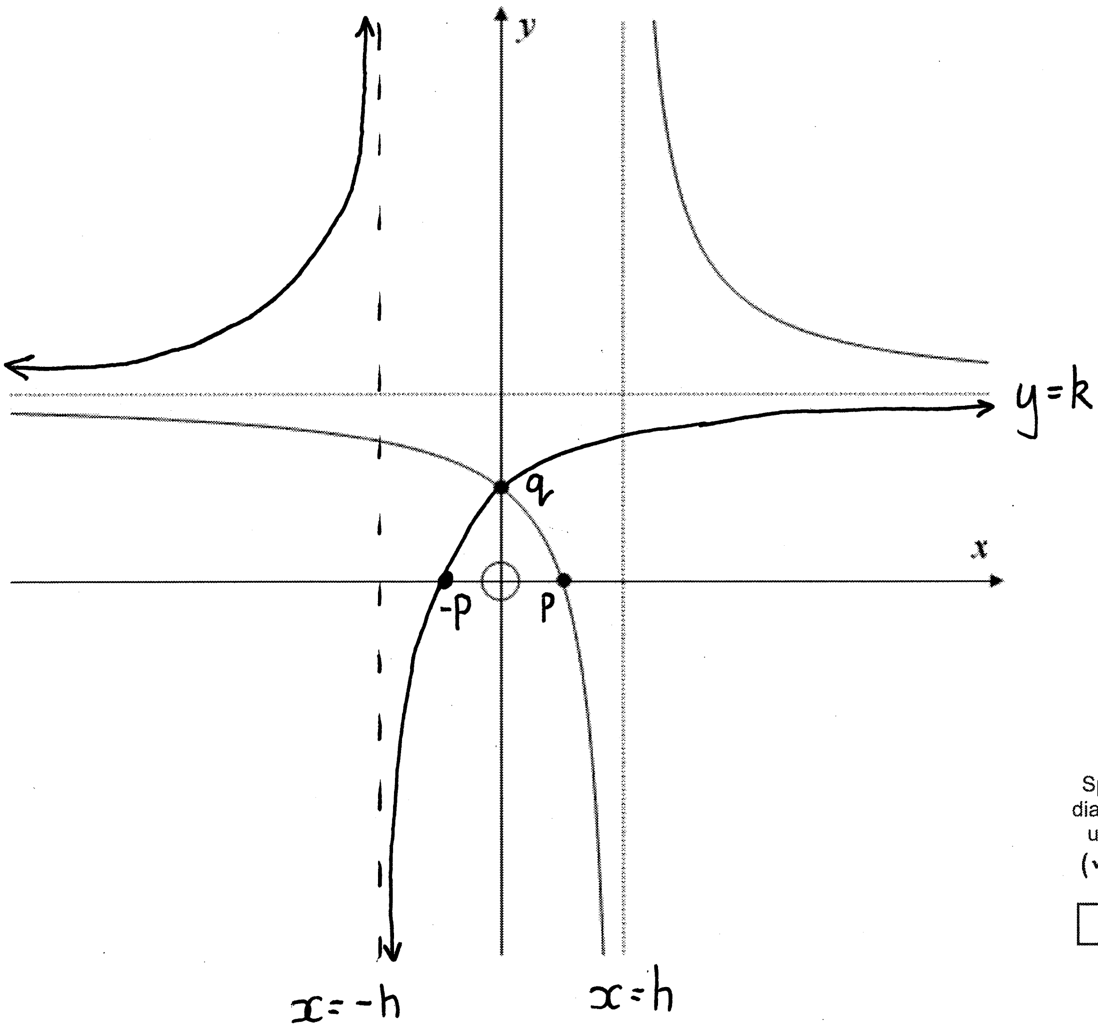
Part 1

- Attempt **all** questions in this part.
- This part assesses **Criterion 4**.

Question 17

A hyperbola of the form $f(x)=\dfrac{a}{x-h}+k$ is sketched below.

The function intersects the x and y axes at p and q respectively.



Spare diagram used (✓) ☐

On the axes above, sketch a graph of $y=f(-x)$.

Label the asymptotes and axes intercepts with the appropriate pronumerals after this graphical transformation.

Marker use

/3

Part 1 continues

Marker use

Question 18

Use log laws to show that $x = 3$ is the only valid solution to the equation
$\log_3(x-2)+\log_3(x+5)=\log_3 8$.

$\log_3 (x-2)(x+5) = \log_3 8$

$\therefore (x-2)(x+5) = 8$

$x^2 + 3x - 10 = 8$

$x^2 + 3x - 18 = 0$

$(x+6)(x-3) = 0$

$\therefore x \neq -6 \quad \text{or} \quad x = 3$

Rejected as $\log(-6-2)$ is undefined

3

Question 19

A square root function and its inverse are defined by:

This should be $x \leq 4$

$f(x) = 4 - 3\sqrt{x}$ and $f^{-1}(x) = \dfrac{(4-x)^2}{9}$ where $x \geq 4$.

Show that $f\{f^{-1}(x)\} = f^{-1}\{f(x)\} = x$.

$f(f^{-1}(x)) = 4 - 3\sqrt{\dfrac{(4-x)^2}{9}}$

$= 4 - 3\left(\dfrac{4-x}{3}\right)$

$= 4 - 4 + x$

$= x$

$f^{-1}(f(x)) = \dfrac{(4-(4-3\sqrt{x}))^2}{9}$

$= \dfrac{(3\sqrt{x})^2}{9}$

$= \dfrac{9x}{9}$

$= x$

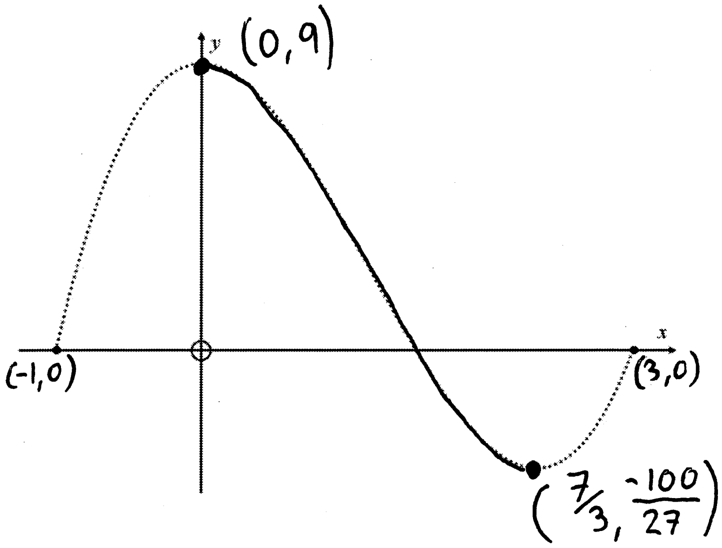
$\therefore f(f^{-1}(x)) = f^{-1}(f(x)) = x$

4

Marker use

Question 20

The function $f(x)=(x+1)(2x-3)(x-3)$ is defined over a restricted domain and sketched below.



a) Determine the **largest** domain so the inverse of $f(x)$ is a function. 2

Solved to find max at $(0,9)$ and min at $\left(7/3, \frac{-100}{27}\right)$

Inverse excists when $f(x)$ is 1:1

$\therefore x \in [0, 7/3]$

b) Hence, determine the domain for this inverse function, $f^{-1}(x)$. 2

Domain of $f^{-1}(x)$ = Range of $f(x)$

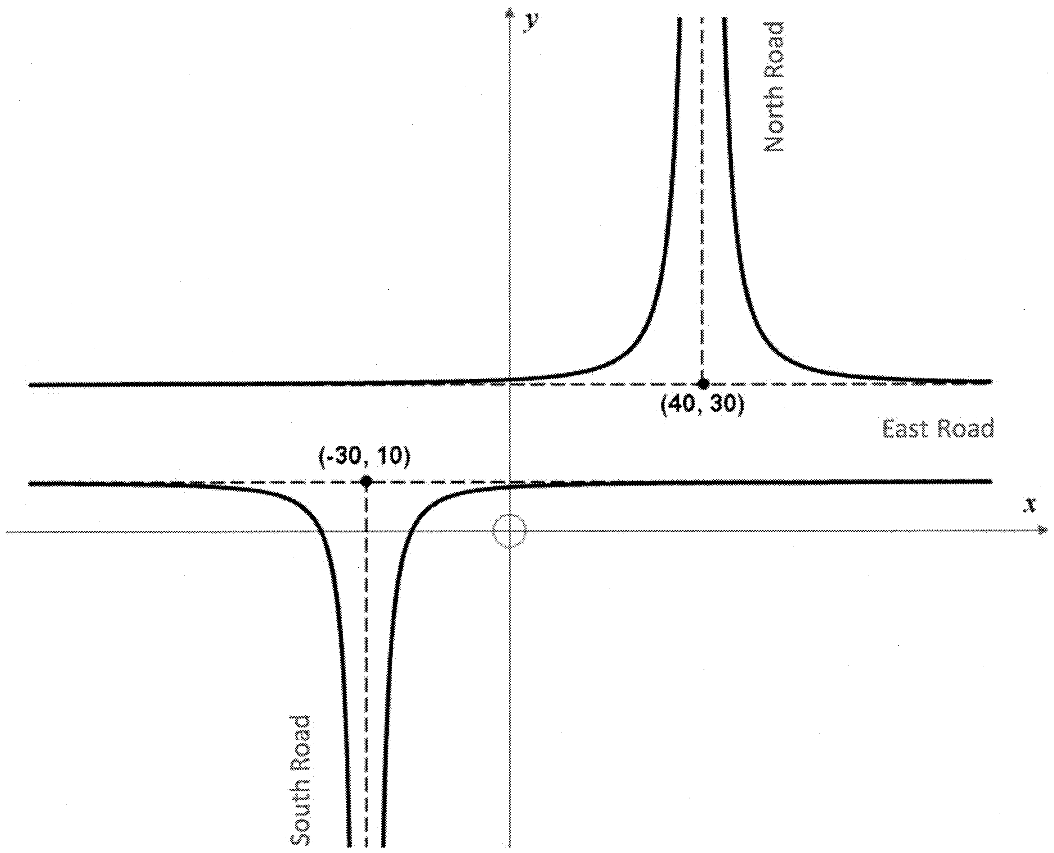
$\therefore x \in \left[\frac{-100}{27}, 9\right]$

Part 1 continued

Marker use

Question 21

$f(x)=\dfrac{1600}{(x-40)^2}+30$ and $g(x)=\dfrac{a}{(x-h)^2}+k$ model roads as shown below.



- Vertical asymptotes model the mid-lines of North Road and South Road.
- Horizontal asymptotes model the width approached by East Road for large distances from the origin.
- All coordinates are measured in metres.

Question 21 continues

Marker use

a) State the width approached by East Road for large distances from the origin.

$30 - 10 = 20m$

/1

In Question 21 b) and c), you may use your calculator to solve any relevant equations.

b) Determine the equation for $g(x)$ given $(-40,1)$ is a point on the function.

Asymptotes give $h = -30$ and $k = 10$

Sub $(-40, 1)$ to find a:

$$1 = \frac{a}{(-40-(-30))^2} + 10$$

$\therefore a = -900$

$$\therefore g(x) = \frac{-900}{(x+30)^2} + 10$$

/2

c) North Road ends 64m above the point $(40,30)$.

Determine the **final width** of North Road.

North Rd ends at $30 + 64 = 94$

Solve $f(x) = 94$

$\therefore x = 35, \quad x = 45$

$\therefore$ Width $= 45 - 35$

$= 10m$

/3

Total C4

/20

Part 2

- Attempt **all** questions in this part.
- This part assesses **Criterion 5**.

Marker use

Question 22

Determine an exact value for $\cos\theta$ given $\sin\theta = \frac{3}{7}$ and $\frac{\pi}{2} < \theta < \pi$.

/3

$\cos^2\theta + \sin^2\theta = 1$

$\cos^2\theta + \left(\frac{3}{7}\right)^2 = 1$

$\cos(\theta) = \pm\sqrt{\frac{40}{49}}$

Since θ is in Q2 :

$= -\frac{\sqrt{40}}{7}$

$= \frac{-2\sqrt{10}}{7}$

Question 23

Consider the trigonometric equation:

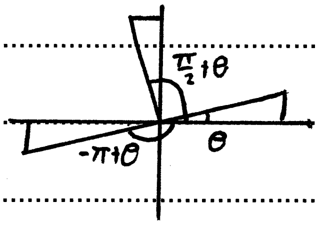
$$\cos\left(\frac{\pi}{2}+\theta\right) + \sin\left(-\pi+\theta\right) = -\frac{1}{2}, \quad \text{where} \quad 0 < \theta < \frac{\pi}{2}.$$

a) Provide appropriate reasoning to show the equation can be simplified to $\sin(\theta) = \frac{1}{4}$.

/3

$-\sin(\theta) + -\sin(\theta) = -\frac{1}{2}$

(Complementary) (Symmetry)



$-2\sin(\theta) = -\frac{1}{2}$

$\therefore \sin(\theta) = \frac{1}{4}$

b) Hence, evaluate a solution for θ in **radians**.

/1

$\theta = \sin^{-1}\left(\frac{1}{4}\right)$

$= 0.253$

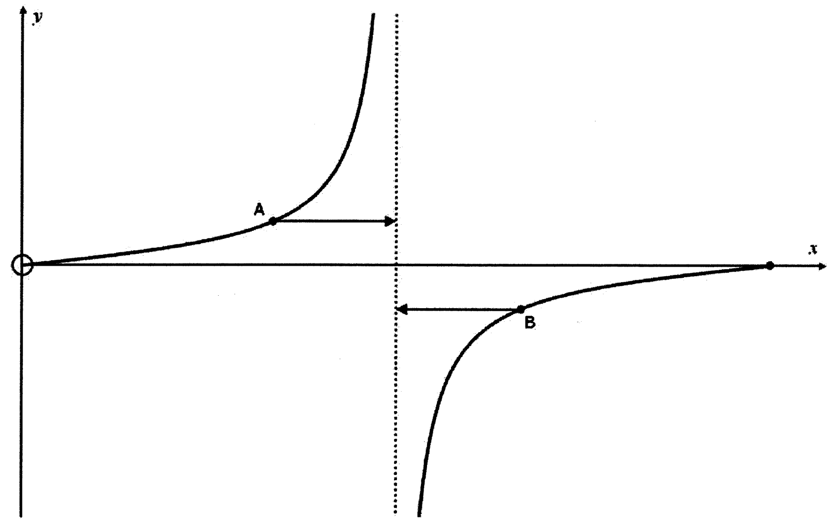
Part 2 continues

Part 2 continued

Marker use

Question 24

The sketch below is for one period of $f(x) = \tan\left(\frac{4\pi x}{3}\right)$.



The two **arrowed lines** each represent a horizontal distance of $\frac{1}{8}$ from points A and B to either side of the asymptote.

Determine the x intercepts, asymptote and hence the coordinates of points A and B.

/5

x ints: $0 = \tan\left(\frac{4\pi x}{3}\right)$

$\therefore x = 0, \; x = 3/4$

Period $= \frac{\pi}{\frac{4\pi}{3}} = 3/4$

Asymptote is halfway between x-ints at $x = 3/8$

A coordinate: $\left(\frac{3}{8} - \frac{1}{8}, f\left(\frac{2}{8}\right)\right) = \left(\frac{1}{4}, \sqrt{3}\right)$

B coordinate: $\left(\frac{3}{8} + \frac{1}{8}, f\left(\frac{4}{8}\right)\right) = \left(\frac{1}{2}, -\sqrt{3}\right)$

Part 2 continues

Part 2 continued

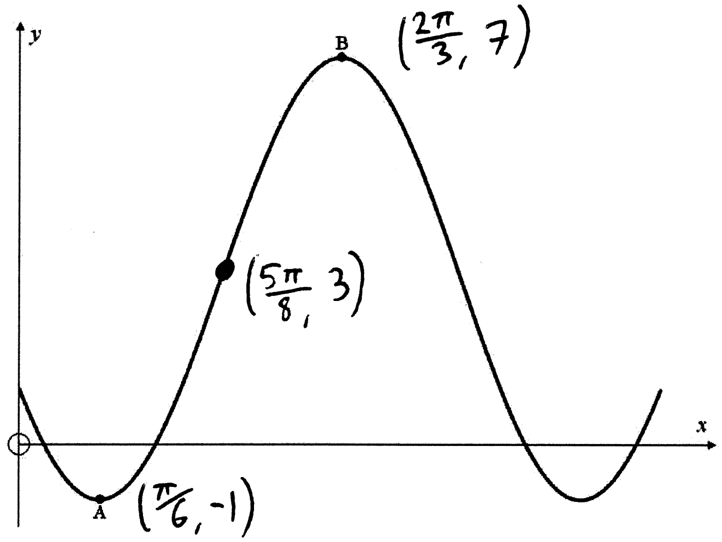
Question 25

Marker use

/ 6

a) Determine an equation of the graph below in the form $y = a\sin\left[n\left(x+b\right)\right]+c$.

The function has a minimum at point A of $\left(\frac{\pi}{6},-1\right)$ and maximum at point B of $\left(\frac{2\pi}{3},7\right)$.



Find c: $\frac{7+-1}{2} = 3 \quad \therefore \; c = 3$

Find a: $\frac{7--1}{2} = 4 \quad \therefore \; a = \pm 4$

Find n: Period $= 2 \times \left(\frac{2\pi}{3} - \frac{\pi}{6}\right)$

$= \pi$

$\therefore \frac{2\pi}{n} = \pi \quad \therefore \; n = \pm 2$

Find b: Sub point between A and B in $\therefore \; b = \frac{5\pi}{12}$

$\therefore \; y = 4\sin\left(2\left(x - \frac{5\pi}{12}\right)\right) + 3$ or $y = 4\sin\left(2\left(x + \frac{7\pi}{12}\right)\right) + 3$

or $y = -4\sin\left(2\left(x + \frac{\pi}{12}\right)\right) + 3$ or $y = -4\sin\left(2\left(x - \frac{11\pi}{12}\right)\right) + 3$

Question 25 continues

Question 25 continued

Marker use

b) There are many possible correct solutions for the equation found in Question 25 a).

Determine the number of possible correct responses for a, n, b and c by circling the appropriate word(s) in the table below.

No working required.

/2

Variable	Number of Possible Correct Responses		
a	One	Two	More than two
n	One	Two	More than two
b	One	Two	More than two
c	One	Two	More than two

Total C5

/20

Part 3

- Attempt **all** questions in this part.
- This part assesses **Criterion 6**.

Question 26

Marker use

A company is producing and selling mathematical tee shirts.

The **profit**, is given by $P(x) = 24x - \frac{x^2}{3} - \left(45 + \frac{x^2}{6} - 25\ln(x)\right)$.

The number of tee shirts made and sold in any given week is denoted by x, where x is greater than zero.

a) Show that $P'(x) = 24 - x + \frac{25}{x}$

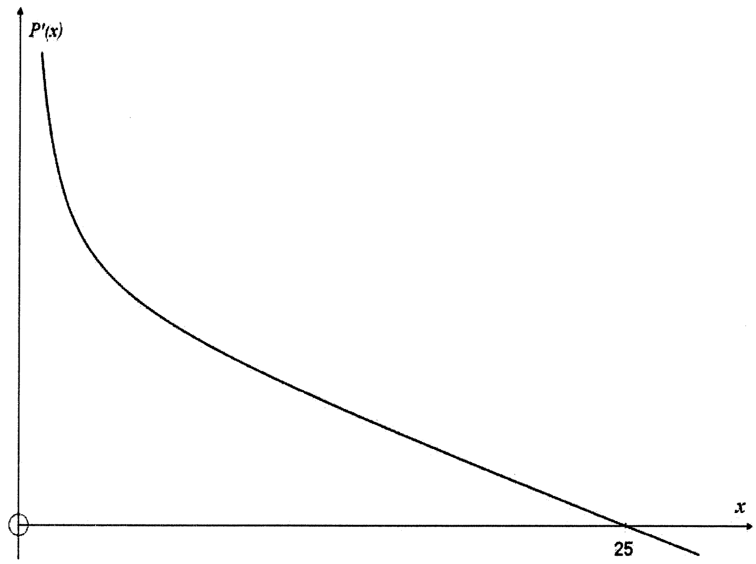
$P(x) = 24x - \frac{x^2}{3} - 45 - \frac{x^2}{6} + 25\ln(x)$

$\therefore P'(x) = 24 - \frac{2x}{3} - \frac{2x}{6} + \frac{25}{x}$

$= 24 - x + 25/x$ as required

/ 2

b) Use the sketch of $P'(x)$ to justify the number of tee shirts required to **maximise** profits.



/ 2

$P'(x) = 0$ at $x = 25$

As $P'(x)$ is +ve when $x < 25$ and -ve when $x > 25$

this is a local maximum.

Part 3 continues

Part 3 continued

Question 27

Marker use

/ 4

Show that the gradient of the function $y = \cos(4x).\tan(2x)$ is decreasing at $x = \frac{\pi}{6}$.

Use the **product** and **chain** rules to provide reasoning.

You may use your calculator to evaluate any substitutions.

$\frac{dy}{dx} = \cos(4x) \cdot 2\sec^2(2x) + -4\sin(4x) \cdot \tan(2x)$

$\frac{d^2y}{dx^2} = 8\cos(4x)\tan^3(2x) - 16\sin(4x)\tan^2(2x) - 8\cos(4x)\tan(2x)$
$- 16\sin(4x)$ using calculator

When $x = \pi/6$, $d^2y/dx^2 = -40\sqrt{3}$
< 0

$\therefore$ The gradient of the function is decreasing at $x = \pi/6$

Part 3 continues

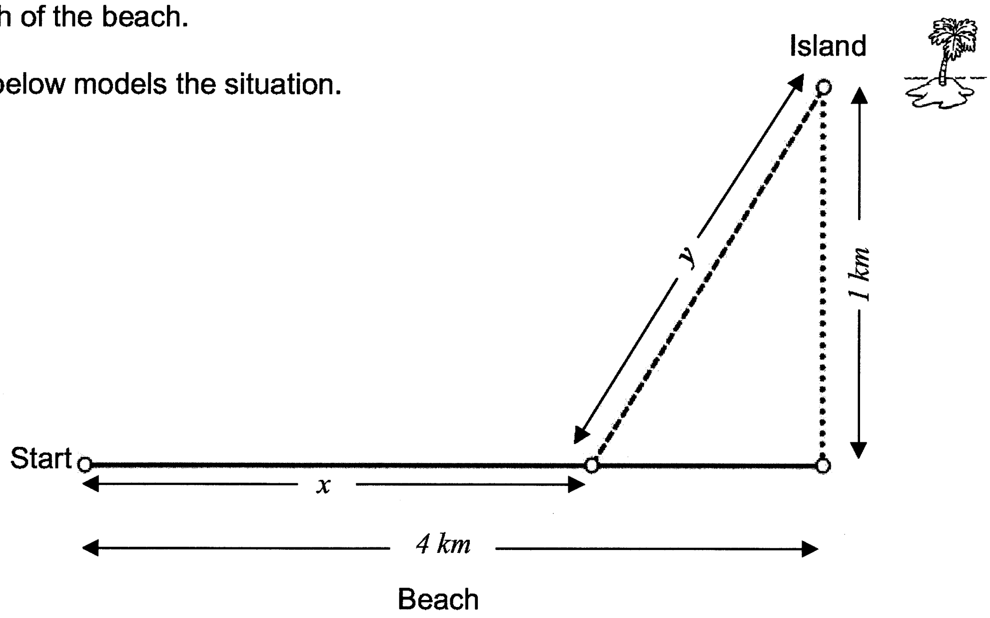
Part 3 continued

Marker use

Question 28

A person starts to walk from one end of a 4 km beach and then swims to an island which is 1 km due north of the beach.

The diagram below models the situation.



They walk at a rate of 5 km/hr and swim at a rate of 3 km/hr.

They want to **minimise** the time taken by walking x km then swimming y km.

The time they take is given by $\mathrm{T}=\frac{x}{5}+\frac{\sqrt{x^2-8x+17}}{3}$ hours.

a) Determine the value for x that minimises the time taken.

There is no need to justify your solution as a minimum.

$\frac{dT}{dx} = \frac{1}{5} + \frac{x-4}{3\sqrt{x^2-8x+17}}$

Solving $dT/dx = 0$ gives $x = 13/4$ or 3.25 hours

/3

b) Hence, determine the minimum time.

Let $x = 13/4$

$\therefore T = 16/15$ or 1.07 hours or 1 hour 4 mins

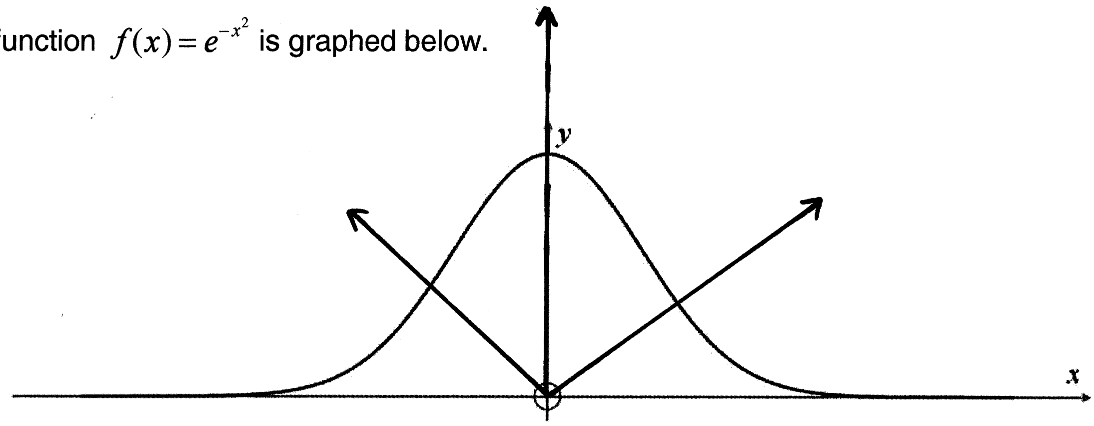
/2

Part 3 continues

Question 29

Marker use

The function $f(x)=e^{-x^2}$ is graphed below.



Spare diagram used (✓)

a) Sketch the **normals** to $f(x)$ that pass through the **origin** on the graph above.

1

b) Show the **gradient** of the **normal** that passes through the point $\left(\frac{\sqrt{\ln 2}}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ equals $\frac{1}{\sqrt{\ln 2}}$.

4

$f'(x) = -2xe^{-x^2}$

$\therefore f'\left(\frac{\sqrt{\ln 2}}{\sqrt{2}}\right) = -2\frac{\sqrt{\ln 2}}{\sqrt{2}} e^{-\left(\frac{\sqrt{\ln 2}}{\sqrt{2}}\right)}$

$= -\sqrt{\ln 2}$

$= m_T$

$m_N = -\frac{1}{m_T} = -\frac{1}{-\sqrt{\ln 2}}$

$= \frac{1}{\sqrt{\ln 2}}$

c) Hence, show that this **normal** passes through the **origin**.

2

Eqn of normal: $y - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{\ln 2}}\left(x - \frac{\sqrt{\ln 2}}{\sqrt{2}}\right)$

$\therefore y = \frac{x}{\sqrt{\ln 2}}$

When $x = 0$, $y = 0$ so it passes through the origin

(or as $c = 0$, y int is 0)

Total C6

/20

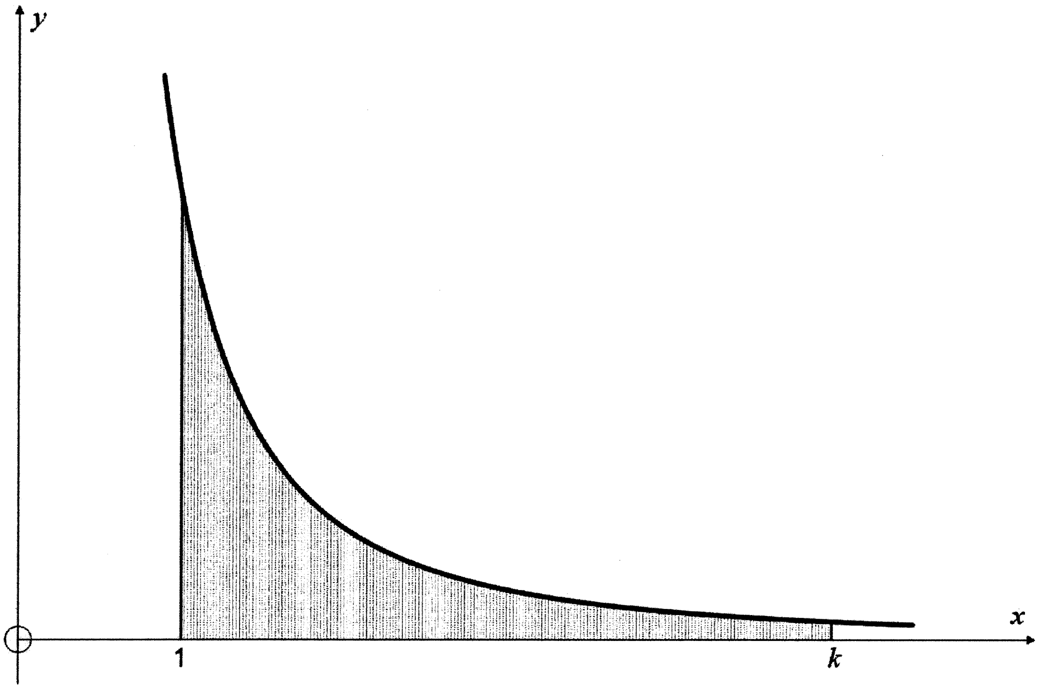
Part 4

- Attempt **all** questions in this part.
- This part assesses **Criterion 7**.

Question 30

Marker use

A section of the function $f(x)=\dfrac{10}{x^2}$ is graphed below.



Determine the value for k where the shaded area equals 8 units^2.

/3

$$\int_1^k f(x)\,dx = 8$$

$$\int_1^k 10x^{-2}\,dx = 8$$

$$\left[\frac{10x^{-1}}{-1}\right]_1^k = 8$$

$$\therefore \quad k = 5$$

Part 4 continues

Part 4 continued

Marker use

Question 31

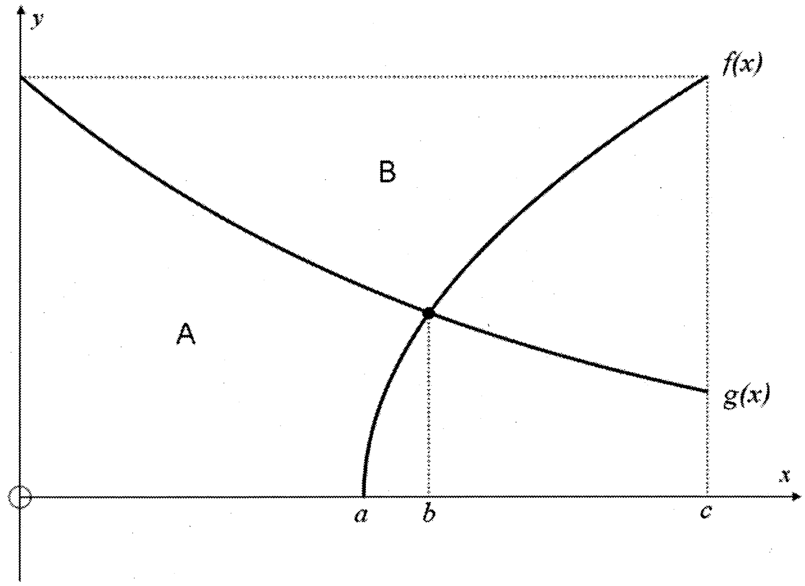
The graph to the right sketches

$f(x)$ for $x \in [a,c]$

and

$g(x)$ for $x \in [0,c]$.

The functions **intersect** where $x = b$.



Determine expressions to evaluate the enclosed areas A and B.

$A = \int_0^a g(x)dx + \int_a^b g(x) - f(x)\,dx$

(units^2)

$B = c \times g(0) - \int_0^b g(x)\,dx - \int_b^c f(x)\,dx$

3

Question 32

The functions $f(x)$ and $g(x)$ are continuous, where:

$f(x) = \frac{1}{4}g(x)$, $\int_0^4 f(x)\mathrm{d}x = -3$ and $\int_4^{10} g(x)\mathrm{d}x = 8$.

Evaluate $\int_0^{10} f(x)\mathrm{d}x$.

$= \int_0^4 f(x)dx + \int_4^{10} f(x)dx$

$= -3 + \int_4^{10} \tfrac{1}{4} g(x)dx$

$= -3 + \tfrac{1}{4}\int_4^{10} g(x)\,dx$

$= -3 + \tfrac{1}{4}(8)$

$= -3 + 2$

$= -1$

3

Part 4 continues

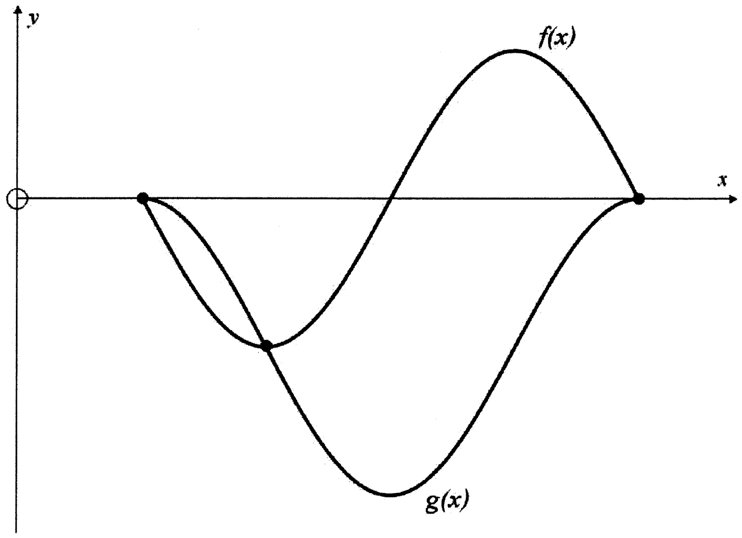
Part 4 continued

Marker use

Question 33

The sketch below has two enclosed areas between the functions:

$f(x) = \cos(3x)$ and $g(x) = \sin(3x) - 1$ over the domain $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$.



Show that the total of the two enclosed areas equals $\frac{4+\pi}{3}$.

/5

Intersections occur when $f(x) = g(x)$

$\therefore x = \pi/6, \pi/3, 5\pi/6$

$\therefore \text{Area} = \int_{\pi/6}^{\pi/3} g(x) - f(x)\, dx + \int_{\pi/3}^{5\pi/6} f(x) - g(x)\, dx$

$= \left[\frac{-\cos(3x)}{3} - x - \frac{\sin(3x)}{3}\right]_{\pi/6}^{\pi/3} + \left[\frac{\sin(3x)}{3} + \frac{\cos(3x)}{3} + x\right]_{\pi/3}^{5\pi/6}$

$= \pi/3 + 4/3$

$= \frac{4+\pi}{3}$ units^2 as required

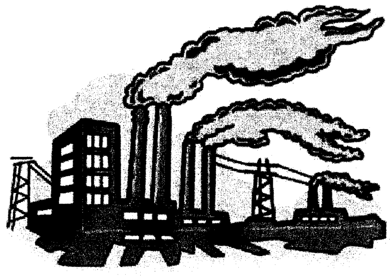
Part 4 continues

Part 4 continued

Marker use

Question 34

A coal-fired power station emits sulfur dioxide.



Concentrations of the gas C, in parts per million (ppm), are monitored at various ground level positions x km away from the site.

The rate of change is modelled by $\frac{\mathrm{d}C}{\mathrm{d}x} = -\frac{1}{2x^3}$.

a) Determine an equation for the concentration given $C = 0$ when $x = 50$.

/ 4

$C = \int \frac{dC}{dx}\, dx$

$= -\frac{1}{2} \int x^{-3}\, dx$

$= \frac{1}{4x^2} + k$ where is k is a constant $(k \in \mathbb{R})$

Sub $(0, 50)$ gives $k = \frac{-1}{10000}$

$\therefore C = \frac{1}{4x^2} - \frac{1}{10000}$

b) Long term human exposure to this gas should not exceed concentrations of 0.03 ppm. Evaluate the minimum distance deemed safe for residents living nearby.

/ 2

Express your answer accurate to the nearest kilometre.

Let $C = 0.03$

$\therefore x = \frac{50}{\sqrt{301}}$

$= 2.882$

≈ 3 km away

Total C7

/ 20

Part 5

- Attempt **all** questions in this part.
- This part assesses **Criterion 8**.

Question 35

Marker use

A discrete random variable X is defined in the following table:

x	-2	0	3
$\Pr(X=x)$	$2a^2 = \frac{1}{8}$	$\frac{5a}{2}$	$a = \frac{1}{4}$

a) Show the only valid solution for the constant a is $\frac{1}{4}$.

$\sum \Pr(X=x) = 1 \quad \therefore \quad 1 = 2a^2 + \frac{5a}{2} + a$

$0 = 4a^2 + 7a - 2$

$\therefore a = -2$ or $a = \frac{1}{4}$

$0 \leq \Pr(X=x) \leq 1$ so reject $a=-2$ as $\Pr(X=-2) = 8$
and $\Pr(X=3) = -2$

/3

b) Hence, determine the variance for the distribution X.

$Var(X) = E(X^2) - [E(X)]^2$

$= (-2)^2(2(\frac{1}{4})^2) + 0 + 3^2(\frac{1}{4}) - [-2(2(\frac{1}{4})^2) + 3(\frac{1}{4})]^2$

$= 4 \times \frac{1}{8} + \frac{9}{4} - [-\frac{1}{4} + \frac{3}{4}]^2$

$= \frac{1}{2} + \frac{9}{4} - [\frac{1}{2}]^2$

$= \frac{11}{4} - \frac{1}{4}$

$= \frac{10}{4}$

$= \frac{5}{2}$ or 2.5

/3

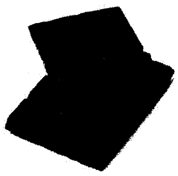
Part 5 continues

Part 5 continued

Question 36

Marker use

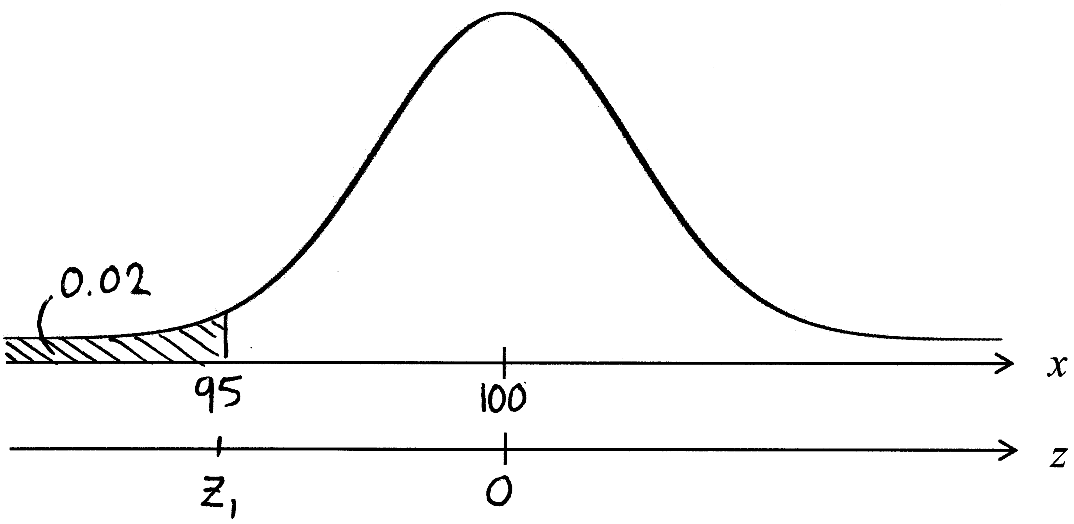
The weight of blocks of a particular brand of chocolate is normally distributed with a mean of 100 grams.



2% of blocks produced weighed under 95 grams and were deemed unacceptable.

a) Show this information on the normal distribution graph below.

/2



b) Show that the standard deviation for the weight of blocks is approximately 2.435 grams.

/4

$\Pr(Z < z_1) = 0.02$

$\therefore z_1 = -2.0537$

Since $z = \dfrac{x-\mu}{\sigma}$

$\sigma = \dfrac{95-100}{-2.0537}$

$= 2.435$ grams

c) Hence, determine the probability of blocks with a weight between 96 and 104 grams.

/1

$X \sim N(100, 2.435)$

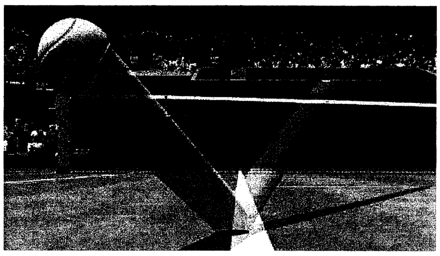
$\Pr(96 < X < 104) = 0.8996$ or 90%

Part 5 continues

Part 5 continued

Marker use

Question 37



An electronic system keeps track of the ball in tennis matches and allows a player to challenge a decision.

A successful challenge leads to a decision being overturned.

A sample of 600 challenges had 168 overturned.

a) Determine the proportion of overturned decisions for this sample. /1

$\hat{p} = \frac{168}{600} = \frac{7}{25}$ or 0.28 or 28%

The table below compares statistics relevant to this sample.

Confidence Interval C%	z_{values}	Margin of Error M
90%	1.645	0.03015
99%	2.576	0.04722

b) Complete the table. Show working below. /3

$Pr(-2.576 < z < 2.576) = 0.99$

$= 99\%$

$M = z\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

$= 1.645\sqrt{\frac{7/25 \cdot 18/25}{600}}$

$= 0.03015$

c) Find and interpret the 90% confidence interval in the context of this tennis scenario. /3

90% CI : $(\hat{p} - M, \hat{p} + M)$

$= (0.28 - 0.03015, 0.28 + 0.03015)$

$= (0.24985, 0.31015)$

$\therefore$ We can be 90% confident that between 25% and 31% challenges are successfully overturned.

Total C8 /20