2021 ASSESSMENT REPORT

MTS 415118 - MATHEMATICS SPECIALISED

Section A
Sequences & Series

This section assess criterion 4

This section was very difficult to complete in the time allotted. It would appear that many trivial errors occurred in the rush to complete this section.

Question 1

a) This was well answered by most students. A common error for $V_{k+1} - V_k$ was not using brackets correctly for $\frac{1}{5}k(k+1)(k+2)(k+3)\{(k+4) - (k-1)\}$.

b) Many students used incorrect notation going from S_n to summing $V_{k+1} - V_k$ with many claiming that $S_n = V_{k+1} - V_k$, however, most students were able to reach the correct answer. A common error was assigning V_1 a value of 24 instead of 0.

Question 2

Very few students achieved full marks for this section.

a) Many students recognised that cos(n) oscillates between -1 and $+1$.

b) Most students recognised that the difference in logs is equivalent to the log of the quotient. Too many students asserted that $\ln(2n+3) - \ln(3n+2)$ diverged to $-\infty$. Another error was to say that it converged to $^2/_3$ instead of $\ln(^2/_3)$.

c) Most students recognised the effect of $(-1)^n$ was to create an oscillation but asserted that arctan diverged to $\pm\infty$.

Question 3

Most students recognised that the common ratio, $r = (x-2)/3$ and that $|r| < 1$ for the sequence to converge. Students who asserted $|r| \leq 1$ lost a mark.

Question 4

a) Most students tried to solve a quadratic equation for n with a minority achieving success due to algebraic errors. Those students who attempted rigging tended to use wrong logic and arrived at an incorrect expression for N(K).

b) Some credit was given to students who substituted K = 100 into their wrong expression. A common mistake was to state that the smallest integer greater than 203.49 was 203.

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Question 5

This was probably the most difficult question in Section A. Most students were able to achieve half marks by showing that proposition, P_1 was true, assuming P_k was true and writing an expression for P_{k+1}. Unfortunately, most students were unable to manipulate the LHS of the equation to have a common denominator of $(1 - x)^2$ and to show how the coefficients of x^k, x^{k+1} and x^{K+2} were obtained. There was a lot of fudging to arrive at the correct answer.

Question 6

The Maclaurin expansion was generally poorly done. Some credit was given for correctly finding the first and second derivatives of the function but less than half of students were able to determine f'(0) and f''(0) and use these to determine the values of m and n.

Section B
Matrices & Linear Transformations

This section assess criterion 5

In general, this section was reasonably well done by most students. The section relied on students being able to manipulate matrices as well as understanding how these matrices could then be used.

Question 7

This was a good starter question and, unsurprisingly, was quite well done. The most common error was confusing the matrices for dilations as opposed to that required for shears and/or having the shear in the wrong direction. Pleasingly, very few students multiplied the matrices in the wrong order. It is important that with any graph that there needs to be an indication of scaling on both axes. There are several ways in which this can be achieved.

Question 8

Again, very well done.

Question 9

Many of the students knew what to do in parts a) and c). Part b) presented problems for many.

Most of this revolved around how to deal with a direction value of 0 (which is what occurred in the *y*-direction).

A number ended up with the statement $\frac{x-3}{1-3} = \frac{y-2}{2-2} = \frac{z-5}{-3-5}$ but were uncertain as to how to proceed from there.

Students who used direction numbers found the going much easier.

Question 10

As in Question 7, most students in part a) successfully derived the correct matrices and then multiplied them in the correct order.

The use of the notation (*X*, *Y*) for the image seemed to affect very few, if any, students.

In part b) most knew what needed to be substituted, however their execution of this was not well done by a number.

Question 11

a) Students who did not use their calculator to reduce the matrix had much more success than those who did. It was possible to reduce the matrix, without using *RREF* form, in three or four steps and to proceed from there. This often presented a clearer path. A number of students were not sure where they needed to include or exclude values.

b) This was reasonably well done by many students.

Question 12

The notion of the trace, which almost all would not have seen, did not present too many problems for part a).

Part b) however required some good insight and reasonable algebra skills and many did not have as much success with this part.

Section C
Differential Calculus + Areas & Volumes

This section assess criterion 6

Overall, this section was a little longer than previous papers, but students demonstrated their differential calculus skills well.

Question 13

a) Almost all students answered this question correctly.

b) Students interpreted $\sin^{-1}$ as arcsin or 1/sin - both were awarded full marks if correct. The majority correctly found the derivative. A few forgot to use the chain rule and did not multiply by the derivative of $x^{1/2}$.

Question 14

The majority of students were able to find the two points of inflection, however, there were quite a few who made algebraic errors. Of concern was the large number of students who did not show that there was change of concavity at these points – this is essential.

Question 15

This was an unfamiliar function with an oblique asymptote, so some allowance was made when awarding marks.

a) Almost all students were able to find the intercepts correctly but many were confused about what critical points actually are! Any stationary points or points of inflection should have been found.

b) This was poorly done, possibly due to the language used in the question.

c) Full marks were awarded for correct shape, critical points and intercepts labelled, and vertical asymptote labelled. Lack of knowledge about the oblique asymptote was not penalised. This question was answered very poorly, which was disappointing when considering that students can draw the graph on their calculator before drawing on paper. Many graphs were drawn as an hyperbola shape which meant that the stationary points discovered in part (a) were not actually there!

Question 16

This question was answered very well by students. Any errors were mostly careless algebraic mistakes.

Question 17

a) This question was reasonably well done. Common errors were incorrect boundaries, forgetting to find the point of intersection of the two functions, and ignoring the y=1 boundary line. Students are strongly advised to draw the graph of the region required which hopefully ensures that they find the key points and therefore the correct region.

b) Again, this was reasonably well done. Many students forgot to subtract the volume between y=1 and the x-axis.

Question 18

It was disappointing that a number of students did not even attempt this question, therefore missing out on a possible 7 marks. Again, drawing the graph of the region is a huge benefit when solving the problem and would have been rewarded with a mark. Of those that did attempt it, many did not understand where the first quadrant is. There was some success when finding the volume of the region when rotated around the x-axis, students struggled with the finding the correct boundaries of the region rotated around the y-axis. Careless algebraic errors were common, these errors were penalised lightly.

Section D
Integral Calculus

This section assess criterion 7

This was a difficult section with a number of unfamiliar questions. Overall, the resilience of students was impressive, with students not dwelling too much on questions they were struggling with, searching for marks wherever they could and not leaving the remainder of a question if they were unable to complete an earlier part.

Question 19

a) Almost all students knew the trigonometric result to apply here. Those that did not usually tried integration by parts with no success. Common errors in this question were forgetting the "+C", using the incorrect trigonometric result and stating that $\int cosxdx = -sinx$. These errors were only penalised lightly.

b) Almost all students successfully used integration by parts here, though some tried to integrate x, rather than differentiating it. A number of students forgot that it was a definite integral. Those that did mostly knew how to apply definite integration to an integration by parts problem. The largest number of errors were from incorrect signs, with these answers being given almost full marks.

Question 20

a) Most students recognised this as a straightforward separable differential equation and arrived at the equation $e^y = x^2 - 4x - C$. Unfortunately, many students wrote the next line as $y = ln(x^2 - 4x) + C$ and lost some marks.

b) Most students did not realise that $ln(x^2 - 4x - 4) > 0$ was needed here, not just $ln(x^2 - 4x - 4) \neq 0$. These students were penalised lightly. Students were given full marks if they carried an incorrect answer from (a) into this part but applied the correct process.

Question 21

Most students coped very well with the messy algebra in this question and knew the process to follow.

a) Almost all students successfully proved this question.

b) All that was required to answer this question was to state that the rate was approaching 0 as

c) $N \to 100$. Some students wrote a great deal but did not directly answer the question.

d) Most students rearranged the equation successfully, though some had $N(100 - N)$ in the numerator, leading to completely different, though no easier, working. Everyone that used partial fractions did so correctly, but many students used their calculator which cost a mark. Students needed to obtain an expression for $N(t)$ to gain full marks. Many lost a mark for finishing early.

e) Not many students provided an answer for this question which was disappointing. Students are reminded that they can be given full marks for an answer here even if they are using an incorrect equation from part (c). Also they could substitute $N = 90$ even if they didn't have an explicit form for $N(t)$ in part (c).

Question 22

This was an easy $v = \frac{y}{x}$ and most students obtained full marks. The most common issue was not solving for y explicitly. Otherwise very well done.

Question 23

a) Only a handful of students answered this question correctly. Most students did not realise they needed to change the limits of integration and even fewer realised the variable of integration could be switched from v to x to give the final answer. A number of students spent a great deal of time exploring this proof and they were given part marks.

b) i. Very few students attempted this question and there were only a handful of correct answers. Some students substituted $x - a$ without realising that a should equal π in this case. Those that got this far usually used their trigonometric identities correctly, however splitting the integral in two was very rarely done.

ii. A number of students attempted this question even if they did not have success with the previous parts. A common error was to try and solve $\int_0^\pi \frac{xsinx}{3+cos^2x}dx$ by mistake and then be left with an x in the numerator after they had substituted for $u = cosx$. Despite this being a long question for 3 marks, some students gained the correct answer.

Question 24

Most students failed to complete this question. Part marks were given for calculating $du = \frac{1}{x^2}dx$ (a number of students said $du = -lnxdx$) and also for arriving at $x = \frac{1}{1-u}$. Very few students were able to arrive at $\int_0^\pi \frac{du}{\sqrt{u}}$ but those that did mostly gained full marks. Students need more practise at changing the limits of integration as this caused problems for some.

Section E
Complex Numbers

This section assess criterion 8

The majority of the students performed well on this section of the paper and showed at least some elements of understanding.

One element of some surprise was that a fair number of students performed much better on Q28 and Q30 than the easier questions earlier in the section. There was some evidence that a few students were pushed for time.

Question 25.

Most students performed satisfactorily; many transformed the complex number z into polar form before taking the cube. The most common error in doing this was to identify the polar angle as $\pi/6$ rather than $-\pi/6$. Some answers didn't make it clear that the student was looking for a result with zero real part.

Question 26.

The expression in part (a) was written in Cartesian form by the overwhelming majority; almost everyone knew they had to multiply by the complex conjugate of the denominator. Most of the lost marks were due to careless arithmetic or algebra.

Too many students used calculators to write the quotient in polar form. Rather than write the (simple) expressions in the numerator and the denominator in separate polar forms and dividing these results, the majority of the answers attempted to directly convert the Cartesian form into the polar equivalent. While they could find the modulus successfully, they were unable to identify the angle without recourse to the calculator.

Those that used the calculator in part (a), rather circumvented the reasoning required to deduce the answer in (b). Some credit was given for students who answered the question this way. Those who had derived the polar result in (a) without calculator tended to be successful in deducing the required form of $\tan(\pi/12)$ in part (b).

Question 27.

The majority of the answers to this question were disappointingly poor. Several didn't know where to start. Of those who wrote the complex numbers as the sum of real and imaginary parts, many were unable to write down expressions for the modulus of z_1+z_2 and z_1-z_2. A wide range of (incorrect) expressions were suggested, and many did not realise what happens when one takes the square of the square-root of an expression! Others tried to proceed by writing z_1 and z_2 in general polar form; whilst it was the easy to write the RHS in an appropriate way, the LHS is much trickier to deal with correctly. Of those that took the polar route, none were successful.

Question 28.

The question was generally well done. Many answers obtained the correct values for α and β; almost all students realised that $P(1-2i)=0$. Incorrect answers arose mainly as a result of careless algebra and slips. Students who performed best on this question evaluated the polynomial using the calculator (as would have been expected). Pleasingly many students knew that if a complex number is a root of a polynomial with real coefficients then the conjugate number is also a root.

The majority then successfully identified a quadratic factor of P(z) and derived the other quadratic factor. Most of the marks lost were due to careless algebra. A few students did not read the question completely and left their answers as a factorised polynomial while they were asked to state the roots of the equation.

Question 29.

The answers to this question were rather mixed. Most realised that one of the inequalities corresponded to a circle; but there was much confusion as to the location of the centre and the radius of this circle. The other inequality was equally confusing; some thought this was another circle rather than region above a straight line; of those who identified that a line was important there were many who couldn't find the line – a popular choice was y=-x rather than the (correct) y=x. Furthermore, the majority of those who found y=x and realised that this is a diameter of the circle, then proceeded to shade the wrong half of the circle. The skills required for this type of question are in need of practice.

Very few good answers to part (b) were given; this is not surprising as this was the trickiest part of the section. The inherent difficulty of this question was not helped by the numerous incorrect attempts at part (a); the regions shaded in (a) meant that the problem in (b) was frequently rendered either trivial or nonsensical.

Question 30.

It was surprising that the most poorly answered part of this question was the first part item. Many students did not seem to understand what was required here; of those who tried this verification, many fudged the algebra or simply wrote down the statement in the question.

In contrast, part (b) was generally well done with a few exceptions. Some answers gave angles lying outside the specified range. Others spoiled their work by supposing that -1 = cis($i\pi/2$) and not cis($i\pi$) thereby introducing errors. Some solutions did not write -1 in polar form at all so answers contained expressions involving the fifth root of (-1).

That said, most who answered part (b) correctly were able to connect the results to part (a) properly to derive the answer. Naturally, once errors had been made in (b), then no progress was made in (c).

SOLUTIONS ON THE FOLLOWING PAGE

Mathematics Specialised 2021

Section A

This section assesses Criterion 4

Section A = 36 marks

1. (a) If

$$v_k = \frac{1}{5}(k-1)k(k+1)(k+2)(k+3),$$

verify that

$$k(k+1)(k+2)(k+3) = v_{k+1} - v_k.$$

(2 marks)

Solution

We observe that

$$v_{k+1} - v_k = \frac{1}{5}\left\{k(k+1)(k+2)(k+3)(k+4) - (k-1)k(k+1)(k+2)(k+3)\right\}$$

(1)

$$= \frac{1}{5}k(k+1)(k+2)(k+3)\left\{(k+4)-(k-1)\right\} = k(k+1)(k+2)(k+3).$$

(1)

Hence $k(k+1)(k+2)(k+3) = v_{k+1} - v_k$ as required.

(b) Evaluate

$$\sum_{k=1}^{n} k(k+1)(k+2)(k+3).$$

(4 marks)

Solution

From part (a) we note that

$$\sum_{k=1}^{n} k(k+1)(k+2)(k+3) = \sum_{k=1}^{n}\left\{v_{k+1} - v_k\right\} = (v_2 - v_1) + (v_3 - v_2) + \cdots + (v_{n+1} - v_n)$$

(2)

$$= v_{n+1} - v_1 = v_{n+1} - 0 = \frac{1}{5}n(n+1)(n+2)(n+3)(n+4).$$ (2)

2. Determine whether the following sequences $\{u_n\}$ converge or diverge.

Give brief justification for your assertions and state the limits of any sequences that do converge.

(a)

$$u_n = \cos(n) \qquad \text{(2 marks)}$$

Solution

As $n \to \infty$, $\cos n$ lies between ± 1, but never settles down. Thus this sequence diverges.

(2)

(b)

$$u_n = \ln(2n+3) - \ln(3n+2) \qquad \text{(3 marks)}$$

Solution

We notice that

$$\ln(2n+3) - \ln(3n+2) = \ln\left(\frac{2n+3}{3n+2}\right) = \ln\left(\frac{2+3/n}{3+2/n}\right) \to \ln\left(\frac{2}{3}\right)$$

as $n \to \infty$.

(2)

Hence the sequence converges, to the limit $\ln(2/3)$.

(1)

(c)

$$u_n = \tan^{-1}\left[(-1)^n\sqrt{n}\right]. \qquad \text{(2 marks)}$$

Solution

We know that for large even values of n, $\tan(u_n)$ is large and positive so that u_n is just less than $\pi/2$.

For large odd values of n, $\tan(u_n)$ is large and negative so that u_n is just greater than $-\pi/2$.

Hence we have an alternating sequence that diverges.

(2)

3. For what values of x does the infinite geometric progression

$$3 + (x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{9}(x-2)^3 + \dots$$

converge?

(4 marks)

Solution

The geometric progression has first term 3 and ratio $r = (x-2)/3$.

(1)

The series converges if and only if $|r| < 1$.

(1)

Hence the series converges if $|x-2| < 3$.

(1)

Hence the GP converges if $1 < x < 5$.

(1)

4. (a) Use formal methods to prove that the sequence

$$\left\{\frac{n^2-10}{2n+7}\right\}$$

diverges to infinity.

(4 marks)

Solution

The sequence $\{u_n\}$ diverges to infinity if, given any K, we can find $N = N(K)$ such that $u_n > K$ whenever $n > N$.

(1)

For sufficiently large N_0 we know that $N_0^2 - 10 > (1/2)N_0^2$ and $2N_0 + 7 < 3N_0$ say.

(1)

Then, as long as $n > N_0$, we have

$$\frac{n^2-10}{2n+7} > \frac{n^2/2}{3n} = \frac{1}{6}n.$$

(1)

Hence if $n > 6K$ (and $n > N_0$) then $u_n > K$.

Thus sequence diverges to infinity.

(1)

(b) Determine the smallest positive integer N for which

$$\frac{N^2-10}{2N+7} > 100.$$

(3 marks)

Solution

If

$$\frac{N^2-10}{2N+7} > 100 \implies N^2 - 10 > 200N + 700 \implies N^2 - 200N > 710 \implies (N-100)^2 > 10710.$$

(1)

Hence

$$N - 100 > \sqrt{10710} = 103.48.. \implies N > 203.48...$$

and the smallest value of N is 204.

(2)

5. Use the method of induction to prove that

$$1 + 2x + 3x^2 + \cdots + nx^{n-1} = \frac{1 - (n+1)x^n + nx^{n+1}}{(1-x)^2} \qquad (x \neq 1).$$

(7 marks)

Solution

First we check expression holds when $n = 1$. Then the right hand side is

$$\frac{1 - 2x + x^2}{(1-x)^2} = 1$$

as required.

(1)

Assume that the expression hold for a particular $n = N$, i.e. assume

$$\sum_{k=1}^{N} kx^{k-1} = \frac{1 - (N+1)x^N + Nx^{N+1}}{(1-x)^2}.$$

(1)

Given this, need to prove the result is true for $n = N + 1$.

(1)

Then

$$\begin{aligned}
\sum_{k=1}^{N+1} kx^{k-1} &= \sum_{k=1}^{N} kx^{k-1} + (N+1)x^N \\
&= \frac{1 - (N+1)x^N + Nx^{N+1}}{(1-x)^2} + (N+1)x^N \\
&= \frac{1 - (N+1)x^N + Nx^{N+1} + (N+1)x^N(1 - 2x + x^2)}{(1-x)^2} \\
&= \frac{1 - (N+1)x^N + Nx^{N+1} + (N+1)x^N - 2(N+1)x^{N+1} + (N+1)x^{N+2}}{(1-x)^2} \\
&= \frac{1 - (N+2)x^{N+1} + (N+1)x^{N+2}}{(1-x)^2} \qquad \text{as required.}
\end{aligned}$$

(3)

Thus if the result holds for $n = N$, it is true for $n = N + 1$.

Since it is true for $n = 1$, holds for all integer n by induction.

(1)

6. The Maclaurin expansion of the function

$$e^{mx} - (1+4x)^n \qquad \text{is} \qquad -4x^2 + \dots.$$

Determine the values of the constants m and n.

(5 marks)

Solution

The Maclaurin expansion of e^{mx} is $e^{mx} = 1 + mx + \frac{1}{2}m^2x^2 + \dots$

(1)

and

$$\begin{aligned}(1+4x)^n &= 1 + n(4x) + \frac{1}{2}n(n-1)(4x)^2 + \dots \\ &= 1 + 4nx + 8n(n-1)x^2 + \dots.\end{aligned}$$

(2)

Hence

$$e^{mx} - (1+4x)^n = (m-4n)x + \left\{\frac{1}{2}m^2 - 8n(n-1)\right\}x^2 + \dots.$$

For the linear coefficient to vanish have $m = 4n$ and then the quadratic term is

$$\frac{1}{2}m^2 - 8n(n-1) = 8n^2 - 8n(n-1) = 8n = -4 \Longrightarrow n = -1/2.$$

(1)

Hence $m = -2$.

(1)

Section B

This section assesses Criterion 5

Section B = 36 marks

7. (a) Sketch the image of the unit square under a shear of size 3 parallel to the y-axis followed by a dilation of factor 4 parallel to the x-axis.

(4 marks)

Solution

The matrix M representing the shear followed by the dilation is

$$\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 3 & 1 \end{pmatrix}.$$

(2)

This matrix takes the points (0,0), (1,0), (0,1) and (1,1) to (0,0), (4,3), (0,1) and (4,4) respectively.

(1)

A neat sketch showing these points.

(1)

(b) What is the area of the image?

(2 marks)

Solution

The determinant of the matrix M is 4

Since areas are multiplied by that factor. Hence the area of the transformed shape is 4 units.

(1)

8. For the matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$$

show that

(a)

$$\mathbf{A^2} = 3\mathbf{A} - 2\mathbf{I}$$

(2 marks)

Solution

$$\mathbf{A}^2 = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 6 & 1 \end{pmatrix}$$

(1)

Then

$$3\mathbf{A} - 2\mathbf{I} = 3\begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 6 & 1 \end{pmatrix} = \mathbf{A}^2$$

as required.

(1)

and (b)

$$2\mathbf{A}^{-1} = 3\mathbf{I} - \mathbf{A}.$$

(2 marks)

Solution

Since the determinant of $\mathbf{A}$ is 2, the inverse

$$\mathbf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ -1 & 1 \end{pmatrix}$$

(1)

Then

$$3\mathbf{I} - \mathbf{A} = 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix} = 2\mathbf{A}^{-1}$$

as required.

(1)

9. Suppose that Q is the plane defined by the equation $4x - 3y + 6z = 36$.

(a) Show that the point A $(3, 2, 5)$ lies on Q.

(1 mark)

Solution

Since $4(3) - 3(2) + 6(5) = 12 - 6 + 30 = 36$, the given point lies on the plane Q.

(1)

(b) Determine the equation of the line that joins A to the point B $(1, 2, -3)$.

(3 marks)

Solution

The direction of the line from B to A is parallel to $(2, 0, 8)$.

(1)

Hence equation of the required line is

$$(x - 3, y - 2, z - 5) = \lambda(2, 0, 8) \qquad \text{or} \qquad \frac{x-3}{2} = \frac{z-5}{8},\ y = 2.$$

(2)

(c) Find the equation of the plane parallel to Q that contains the point B.

(2 marks)

Solution

The desired form of the plane must be given by $4x - 3y + 6z = K$ for some K.

(1)

As B lies on plane then $K = 4(1) - 3(2) + 6(-3) = 4 - 6 - 18 = -20$.

(1)

Hence required plane is $4x - 3y + 6z = -20$.

10. A curve C is rotated by $\pi/4$ **clockwise** about the origin and then reflected in the line $y = x/\sqrt{3}$.

(a) Show that if the point $x, y)$ is transformed to (X, Y) then

$$(1+\sqrt{3})y - (\sqrt{3}-1)x = 2\sqrt{2}X \qquad \text{and} \qquad (1+\sqrt{3})x + (\sqrt{3}-1)y = 2\sqrt{2}Y.$$

(4 marks)

Solution

The matrices for a rotation through $\pi/4$ clockwise is and reflection in $y = x/\sqrt{3}$ are

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad \text{and} \quad \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \qquad \text{respectively.}$$

(2)

If the point (x, y) is transformed to (X, Y) then

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} x+y \\ y-x \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}}\begin{pmatrix} x(1-\sqrt{3}) + y(1+\sqrt{3}) \\ (1+\sqrt{3})x + (\sqrt{3}-1)y \end{pmatrix},$$

Hence

$$(1+\sqrt{3})y - (\sqrt{3}-1)x = 2\sqrt{2}X \qquad \text{and} \qquad (1+\sqrt{3})x + (\sqrt{3}-1)y = 2\sqrt{2}Y.$$

(2)

(b) If the equation of the image curve is $xy = 1$, determine the equation of C.
Solution

Now the image curve is $XY = 1$ which, in terms of the original variables, is

$$\left[(1+\sqrt{3})x + (\sqrt{3}-1)y\right]\left[(1+\sqrt{3})y - (\sqrt{3}-1)x\right] = 8$$

(1)

Coefficient of x^2 is $-(1+\sqrt{3})(\sqrt{3}-1) = -2$.

Coefficient of y^2 is $(\sqrt{3}-1)(\sqrt{3}+1) = 2$.

Coefficient of xy is $(1+\sqrt{3})^2 - (\sqrt{3}-1)^2 = 4\sqrt{3}$.

(1)

What remains is $2y^2 - 2x^2 + 4\sqrt{3}xy = 8$ or $y^2 - x^2 + 2\sqrt{3}xy = 4$. which is the equation of C.(1)

11. A system of linear equations for three unknowns is given by

$$\begin{aligned} x - 2y + z &= 7 \\ 2x + y - 2z &= 1 \\ -x + \alpha y + 2z &= \beta. \end{aligned}$$

(a) What are the restrictions on α and β if this system has no solution?

(5 marks)

Solution

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 7 \\ 2 & 1 & -2 & 1 \\ -1 & \alpha & 2 & \beta \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 7 \\ 0 & 5 & -4 & -13 \\ 0 & \alpha-2 & 3 & \beta+7 \end{array}\right) \qquad (1)$$

$$\Longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 7 \\ 0 & 5 & -4 & -13 \\ 0 & 0 & 3-\frac{4}{5}(2-\alpha) & \beta+7-\frac{13}{5}(2-\alpha) \end{array}\right) \qquad (1)$$

$$\Longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 7 \\ 0 & 5 & -4 & -13 \\ 0 & 0 & 4\alpha+7 & 5\beta+13\alpha+9 \end{array}\right) \qquad (1)$$

This system will have a unique solution if $\alpha \neq -7/4$.

(1)

Suppose $\alpha = -7/4$. Now if the right hand side of the last equation happens to be zero ($\beta = 11/4$) then this equation is trivial and the system still admits a solution.

Hence we require both $\alpha = -7/4$ and $\beta \neq 11/4$ for the system to have no solution.

(1)

(b) If the system has the unique solution $(x, y, z) = (3, -1, 2)$ determine the relationship between α and β.

(2 marks)

Solution

Clearly the solution $(3, 1, 2)$ satisfies the first two equations. It satisfies the third equation if $\alpha + \beta = 1$.

(1)

BUT if $\alpha = -7/4$ we have already seen the system admits either infinitely many or no solutions.

So for a unique solution we have have both $\alpha + \beta = 1$ and $\alpha \neq -7/4$.

(1)

12. Let the matrix

$$\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in which a, b, c and d are real numbers. The **trace** of $\mathbf{M}$, written $Tr(\mathbf{M})$, is defined to be the sum of the leading diagonal entries so $Tr(\mathbf{M}) = a + d$.

(a) Prove that

$$Tr(\mathbf{M}^2) = [Tr(\mathbf{M})]^2 - 2\det(\mathbf{M}).$$

(3 marks)

Solution

Now

$$\mathbf{M}^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & bc + d^2 \end{pmatrix} \qquad (\ddagger)$$

so that $Tr(\mathbf{M}^2) = a^2 + d^2 + 2bc$.

(2)

Also $[Tr(\mathbf{M})]^2 - 2\det(\mathbf{M}) = (a+d)^2 - 2(ad - bc) = a^2 + d^2 + 2bc$

(1)

so $Tr(\mathbf{M}^2) = [Tr(\mathbf{M})]^2 - 2\det(\mathbf{M})$ as required.

(b) If $\mathbf{M}^2 = \mathbf{I}$, but $\mathbf{M} \neq \pm\mathbf{I}$, find the trace and determinant of $\mathbf{M}$.

(3 marks)

Solution

If $\mathbf{M}^2 = \mathbf{I}$ the off-diagonal elements of $\mathbf{M}^2$ must be zero. From equation ($\ddagger$) this means that $b(a+d) = c(a+d) = 0$.

(1)

Two cases can occur; either $a + d = 0$ or $b = c = 0$.

If $b = c = 0$ then if $\mathbf{M}^2 = \mathbf{I}$ then $a^2 = d^2 = 1$ and $a = -d = \pm 1$ (if $a = d = \pm 1$ then $\mathbf{M} = \pm\mathbf{I}$ that is excluded). Then the trace of $\mathbf{M}$ is 0 and the determinant is -1.

The alternative is that $a + d = 0$. then immediately we have that the trace of $\mathbf{M}$ is 0 and the result of part (a) gives the determinant if -1.

In both eventualities we conclude that $\mathbf{M}$ has zero trace and determinant -1.

(2)

Section C

This section assesses Criterion 6

Section C = 36 marks

13. (a) Differentiate $x^4 e^{-2x}$ with respect to x.

(2 marks)

Solution

Use the product rule to give

$$\frac{d}{dx}(x^4 e^{-2x}) = 4x^3 e^{-2x} + x^4(-2e^{-2x}) = 2x^3(2-x)e^{-2x}.$$

(2)

(b) Determine

$$\frac{d}{dx}\left\{\sin^{-1}(\sqrt{x})\right\}.$$

(2 marks)

Solution

Since

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

then

$$\frac{d}{dx}\left\{\sin^{-1}(\sqrt{x})\right\} = \frac{1}{\sqrt{1-(\sqrt{x})^2}}\frac{d}{dx}(\sqrt{x}) = \frac{1}{2}x^{-1/2}\frac{1}{\sqrt{(1-x)}} = \frac{1}{2\sqrt{x(1-x)}}.$$

(2)

14. Determine the two points of inflection of the function

$$y(x) = e^{x(2-x)}.$$

(4 marks)

Solution

If $y = e^{x(2-x)}$ then

$$y' = 2(1-x)e^{x(2-x)}$$

(1)

and

$$y'' = \left[4(1-x)^2 - 2\right] e^{x(2-x)}.$$

(1)

Points of inflection occur where $y'' = 0$ so $(1-x)^2 = 1/2 \implies x = 1 \pm (1/\sqrt{2})$.

(1)

Now

$$x(2-x) = 1 - (x-1)^2 = 1/2$$

so the points of inflection are

$$\left(1 \pm \frac{1}{\sqrt{2}}, \sqrt{e}\right).$$

(1)

15. Consider the function

$$f(x) = \frac{x^2 - 4x + 20}{x - 2}.$$

(a) Determine the intercepts and critical points of $f(x)$.

(4 marks)

Solution

We have $f(0) = -10$ and there are no real values x for which $f = 0$.

(1)

Observe that

$$f(x) = \frac{(x-2)^2 + 16}{x - 2} = x - 2 + \frac{16}{x-2}.$$

Then $f'(x) = 1 - 16(x-2)^{-2}$ and $f''(x) = 32(x-2)^{-3}$.

Then $f' = 0$ when $x - 2 = \pm 1/4$ so $x = 7/4$ and $x = 9/4$.

Now at $x = 7/4$ have $f'' < 0$ so this is a maximum.

At $x = 9/4$ have $f'' > 0$ so a mimimum. There are no points of inflection.

(3)

(b) Find the behaviour of $f(x)$ as $x \to \pm\infty$.

(1 mark)

Solution

Since $f(x) = x - 2 + 16/(x-2)$ then $f(x) \to x - 2$ as $x \to \pm\infty$.

(1)

(c) Sketch the graph of $y = f(x)$.

(4 marks)

Solution

We need a neat sketch of the curve with the following features:

- Verical asymptote at $x = 2$
- Cutting the vertical axis at -10.
- Shows correct asymptotic behaviour as $x \to \pm\infty$.
- Shows a maximum to the left of the vertical asymptote and a minimum to the right.
- Indicates correct behaviour at $x \to 2$.

(4)

16. Consider the curve given implicitly by

$$(x^2 + y^2)^3 = 8x^2y^2.$$

Determine the equations of the tangent and normal at the point (1,1).

(5 marks)

Solution

If we have

$$(x^2 + y^2)^3 = 8x^2y^2$$

then implicit differentiation gives that

$$3(x^2 + y^2)^2(2x + 2yy') = 8[2xy^2 + 2x^2yy'] \qquad (\dagger)$$

(2)

At $x = y = 1$ then $24(1 + y') = 16(1 + y')$ so $y' = -1$.

(1)

Thus the required tangent is

$$y - 1 = -1(x - 1)$$

or $y = 2 - x$.

(1)

The normal is $y - 1 = (x - 1)$ or $y = x$.

(1)

17. (a) Determine the area of the region R enclosed by the parabola $y = x^2$, the hyperbola $y = 8/x$ and the line $y = 1$.

(4 marks)

Solution

The parabola meets the hyperbola at $x = 2$, the hyperbola meets the line at $x = 8$ and the parabola meets the line at $x = 1$.

(1)

Hence the desired area is

$$\begin{aligned} A &= \int_1^2 \left(x^2 - 1\right)\, dx + \int_2^8 \left(\frac{8}{x} - 1\right)\, dx \qquad (1) \\ &= \left[\frac{1}{3}x^3 - x\right]_1^2 + \left[8\ln x - x\right]_2^8 \\ &= \frac{8}{3} - 2 - \frac{1}{3} + 1 + 8\ln 8 - 8 - 8\ln 2 + 2 \\ &= 8\ln 4 - \frac{14}{3} \\ &= 16\ln 2 - \frac{14}{3}. \end{aligned}$$

(2)

(b) Determine the volume generated if R is rotated about the x-axis.

(3 marks)

Solution

The required volume is

$$\begin{aligned} V &= \pi\int_1^2 \left(x^2\right)^2\, dx + \pi\int_2^8 \left(\frac{8}{x}\right)^2\, dx - \pi\int_1^8 (1)^2 dx \\ &= \pi\left[\frac{1}{5}x^5\right]_1^2 + 64\pi\left[-\frac{1}{x}\right]_2^8 - 7\pi \\ &= \frac{31}{5}\pi + 24\pi - 7\pi \\ &= \frac{116}{5}\pi. \end{aligned}$$

(3)

18. Given that μ is a positive constant, the area A is the region in the first quadrant bounded by the parabola $y = \mu(4 - x^2)$ and the co-ordinate axes.

The area A is rotated about the x-axis to form a solid of volume V_1.

The same area is next rotated about the y-axis to generate another solid of volume V_2.

If $V_1 = V_2$ find the value of μ.

(7 marks)

Solution

If the area is rotated about the x-axis to form V_1 then

$$V_1 = \pi \int_0^2 y^2\,dx = \pi\mu^2 \int_0^2 (4 - x^2)^2\,dx = \pi\mu^2 \int_0^2 [16 - 8x^2 + x^4]\,dx$$

$$= \pi\mu^2 \left[16x - \frac{8}{3}x^3 + \frac{1}{5}x^5\right]_0^2 = \pi\mu^2 \left[32 - \frac{64}{3} + \frac{32}{5}\right] = 32\pi\mu^2 \left[1 - \frac{2}{3} + \frac{1}{5}\right] = \frac{2^8}{15}\mu^2\pi.$$

(3)

If the area is rotated about the y-axis then

$$V_2 = \pi \int_0^{4\mu} x^2\,dy$$

where $x^2 = 4 - (y/\mu)$.

(1)

It follows that

$$V_2 = \pi \int_0^{4\mu} \left(4 - \frac{y}{\mu}\right)\,dy = \pi\left[4y - \frac{y^2}{2\mu}\right]_0^{4\mu} = 8\pi\mu.$$

(2)

If $V_1 = V_2$ then

$$\frac{2^8}{15}\mu^2 = \frac{8}{\mu} \implies \mu = 32/15.$$

(1)

Section D

This section assesses Criterion 7

Section D = 36 marks

19. (a) Determine

$$\int \cos 3\theta \cos 2\theta \, d\theta$$

(3 marks)

Solution

Since $2\cos 3\theta \cos 2\theta = \cos 5\theta + \cos\theta$ then

(1)

$$\int \cos 3\theta \cos 2\theta \, d\theta = \frac{1}{2}\int (\cos 5\theta + \cos\theta)\, d\theta = \frac{1}{10}\sin 5\theta + \frac{1}{2}\sin\theta + C.$$

(2)

(b) Evaluate

$$\int_0^1 xe^{-2x}\, dx\,.$$

(3 marks)

Solution

Integrating by parts gives that

$$\int_0^1 xe^{-2x}\, dx = \left[-\frac{1}{2}xe^{-2x}\right]_0^1 + \frac{1}{2}\int_0^1 e^{-2x}\, dx$$

(2)

$$= -\frac{1}{2}e^{-2} - \left[\frac{1}{4}e^{-2x}\right]_0^1 = \frac{1}{4} - \frac{3}{4}e^{-2}.$$

(1)

20. (a) Solve

$$\frac{dy}{dx} = e^{-y}(2x-4), \qquad y(5) = 0\,.$$

(3 marks)

Solution

The equation is separable so that

$$\int e^{y}\,dy = \int (2x-4)\,dx$$

(1)

Then $e^{y} = x^2 - 4x + c$.

(1)

Since $y(5) = 0$ then $1 = 25 - 20 + c \rightarrow c = -4$.

Thus $e^{y} = x^2 - 4x - 4$ or $y = \ln(x^2 - 4x - 4)$.

(1)

(b) For what values of x is your solution valid?

(2 marks)

Solution

Now the right-hand side needs to be positive so $(x-2)^2 - 8 > 0$ implying that $x > 2 + \sqrt{8}$ or $x < 2 - \sqrt{8}$.

(1)

Since we are given the data at $x = 5$, our solution is valid for $x > 2 + \sqrt{8}$.

(1)

21. On a certain nature reserve there are initally 10 pairs of nesting black cockatoos.

One theory suggests that the number N, of nesting pairs after t years will satisfy the differential equation

$$\frac{dN}{dt} = \frac{N}{180}(100 - N).$$

(a) Show that initially the rate of increase of N is 5 per year.

(1 mark)

Solution

When $N = 10$ then $dN/dt = 10 \times 90/180 = 5$. Hence initial rate of increase is 5 per year.

(1)

(b) Interpret what happens as $N \to 100$.

(1 mark)

Solution

As N approaches 100 then $dN/dt \to 0$ meaning that the rate of increase falls to zero. Thus the population is limited to the value 100.

(1)

(c) Determine $N(t)$.

(4 marks)

Solution

$$\frac{dN}{dt} = \frac{N}{180}(100 - N) \Longrightarrow \int \frac{dN}{N(100 - N)} = \int \frac{dt}{180}. \qquad (1)$$

By partial fractions

$$\int \frac{dN}{N(100 - N)} = \frac{1}{100} \int \left(\frac{1}{N} + \frac{1}{100 - N} \right) dN = \frac{1}{100} [\ln N - \ln(100 - N)] = -\frac{1}{100} \ln \left(\frac{100 - N}{N} \right).$$

(1)

Thus

$$\ln \left(\frac{100 - N}{N} \right) = -\frac{5}{9} t + K$$

and $N = 10$ when $t = 0$ implies $K = \ln 9$. Then

$$\frac{100 - N}{N} = 9e^{-5t/9} \Longrightarrow N = \frac{100}{1 + 9\exp(-5t/9)}. \qquad (2)$$

(d) After how many years does the population of black cockatoos reach 90% of its maximum?

(2 marks)

Solution

When $N = 90$ then

$$1 + 9\exp(-5t/9) = 10/9 \Longrightarrow \exp(-5t/9) = \frac{1}{81} \Longrightarrow t = 7.91$$

Hence the 90% population is raeached after nearly 8 years.

(2)

22. Solve the differential equation

$$x\frac{dy}{dx} = y + \frac{y^2}{x}, \qquad y(1) = 1.$$

(5 marks)

Solution

Put $y = vx$ so $y' = xv' + v$

(2)

and

$$xv' + v = v + v^2 \Longrightarrow x\frac{dv}{dx} = v^2.$$

(1)

Hence

$$\int \frac{dv}{v^2} = \int \frac{dx}{x} \Longrightarrow -\frac{1}{v} = \ln|x| + C.$$

Now $v = 1$ when $x = 1$ so $C = -1$.

(1)

Thus

$$-\frac{x}{y} = \ln x - 1 \qquad \text{or} \qquad y = \frac{x}{1 - \ln x}.$$

(1)

23. (a) Use the substitution $v = a - x$ to prove that

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

(2 marks)

Solution

If $u = a - x$ then when $x = 0, u = a$ and when $x = a, u = 0$ and $du/dx = -1$.

Then

$$\int_0^a f(x)\,dx = -\int_a^0 f(a-u)\,du = \int_0^a f(a-u)\,du = \int_0^a f(a-x)\,dx.$$

(2)

Define the integral

$$I = \int_0^\pi \frac{x\sin x}{3+\cos^2 x}\,dx.$$

(b) (i) Use the result of (a) to demonstrate that

$$2I = \pi\int_0^\pi \frac{\sin x}{3+\cos^2 x}dx.$$

(2 marks)

Solution

By part (a),

$$I = \int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{3+\cos^2(\pi-x)}dx = \int_0^\pi \frac{(\pi-x)\sin x}{3+(-\cos x)^2}dx$$

$$= \int_0^\pi \frac{(\pi-x)\sin x}{3+\cos^2 x}dx = \int_0^\pi \frac{\pi\sin x}{3+\cos^2 x}dx - I \Longrightarrow 2I = \pi\int_0^\pi \frac{\sin x}{3+\cos^2 x}dx.$$

(2)

(ii) Hence, or otherwise, evaluate I.

(3 marks)

Solution

If $v = \cos x$ then when $x = 0$, $v = 1$ and when $x = \pi$, $v = -1$. Also $dv/dx = -\sin x$ so

$$2I = \pi\int_{-1}^1 \frac{dv}{3+v^2} = \frac{\pi}{\sqrt{3}}\left[\tan^{-1}\left(\frac{v}{\sqrt{3}}\right)\right]_{-1}^1 = \frac{\pi}{\sqrt{3}}\left[\frac{\pi}{6}+\frac{\pi}{6}\right] = \frac{\pi^2}{3\sqrt{3}} \Longrightarrow I = \frac{\pi^2}{6\sqrt{3}}.$$

(3)

24. Use the substitution $u = 1 - (1/x)$ to evaluate

$$\int_{9/8}^{4/3} \frac{dx}{x^{3/2}\sqrt{x-1}}\,.$$

(5 marks)

Solution

If $u = 1 - (1/x)$ then $du/dx = 1/x^2$, $x = 1/(1-u)$ and $x - 1 = u/(1-u)$

(2)

Then

$$\begin{aligned}
\int_{9/8}^{4/3} \frac{dx}{x^{3/2}\sqrt{x-1}} &= \int_{1/9}^{1/4} x^{1/2} \frac{du}{\sqrt{x-1}} \\
&= \int_{1/9}^{1/4} \frac{1}{\sqrt{1-u}} \sqrt{\frac{1-u}{u}}\, du \\
&= \int_{1/9}^{1/4} u^{-1/2}\, du
\end{aligned}$$

(2)

$$\begin{aligned}
&= 2\left[u^{1/2}\right]_{1/9}^{1/4} \\
&= 2\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{3}.
\end{aligned}$$

(1)

Section E

This section assesses Criterion 8

Section E = 36 marks

25. Show that if $z = \sqrt{3} - i$ then z^3 is purely imaginary.

(3 marks)

Solution

Now $z^2 = (\sqrt{3} - i)(\sqrt{3} - i) = 2 - 2\sqrt{3}i$

(1)

and $z^3 = 2(1 - \sqrt{3}i)(\sqrt{3} - i) = 2(\sqrt{3} - i - \sqrt{3} - 3i) = -8i$

(1)

Hence z^3 purely imaginary.

(1)

26. (a) Express the complex number

$$w = \frac{1+i}{\sqrt{3}+i}$$

in both Cartesian and polar forms.

(4 marks)

Solution

In Cartesian form

$$w = \frac{1+i}{\sqrt{3}+i} = \frac{(1+i)(\sqrt{3}-i)}{(\sqrt{3}+i)(\sqrt{3}-i)} = \frac{(\sqrt{3}+1)+i(\sqrt{3}-1)}{3+1} = \frac{1}{4}[(\sqrt{3}+1)+(\sqrt{3}-1)i.].$$

(2)

In polar form

$$w = \frac{1+i}{\sqrt{3}+i} = \frac{\sqrt{2}\exp(i\pi/4)}{2\exp(i\pi/6)} = \frac{1}{\sqrt{2}}\exp(i\pi/12).$$

(2)

(b) Hence, or otherwise, deduce an exact value for $\tan(\pi/12)$.

(3 marks)

Solution

Comparing the real and imaginary parts of the two forms of the complex number w we see that if $w = R\exp(i\varphi)$ then

$R\cos(\pi/12) = (\sqrt{3}+1)/4$ and $R\sin(\pi/12) = (\sqrt{3}-1)/4$.

(2)

Dividing gives

$$\tan(\pi/12) = \frac{\sqrt{3}-1}{\sqrt{3}+1} \ (= 2-\sqrt{3}).$$

(1)

27. If z_1 and z_2 are any two complex numbers prove that

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left\{|z_1|^2 + |z_2|^2\right\}$$

.

(3 marks)

Solution

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$\begin{aligned}
|z_1 + z_2|^2 + |z_1 - z_2|^2 &= |(x_1 + x_2) + i(y_1 + y_2)|^2 + |(x_1 - x_2) + i(y_1 - y_2)|^2 \\
&= (x_1 + x_2)^2 + (y_1 + y_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 \\
&= 2\left\{x_1^2 + x_2^2 + y_1^2 + y_2^2\right\} \\
&= 2\left\{x_1^2 + y_1^2\right\} + 2\left\{x_2^2 + y_2^2\right\} \\
&= 2\left\{|z_1|^2 + |z_2|^2\right\}
\end{aligned}$$

(3)

28. (a) Determine the real numbers α and β if $z - 1 + 2i$ is one factor of

$$p(z) = z^4 + \alpha z^3 + 18z^2 + \beta z + 35$$

(4 marks)

Solution

If $z - 1 + 2i$ is a factor of $p(z)$ then $p(1 - 2i) = 0$.

(1)

If $v = 1 - 2i$ then $v^2 = -3 - 4i$, $v^3 = -11 + 2i$ and $v^4 = -7 + 24i$.

(1)

Then

$$p(v) = (-7 + 24i) + \alpha(-11 + 2i) + 18(-3 - 4i) + \beta(1 - 2i) + 35 = 0.$$

Real parts give $-11\alpha + \beta = 26$ and imaginary parts give $\alpha - \beta = 24$.

Adding gives $\alpha = -5$ which then gives $\beta = -29$.

(2)

(b) Hence, or otherwise, determine all the solutions of $p(z) = 0$.

(4 marks)

Solution

If $z = 1 - 2i$ is a solution of $p(z) = 0$ then so if $z = 1 + 2i$.

(1)

A quadratic factor of $p(z)$ is therefore $(z - (1 - 2i))(z - (1 + 2i)) = z^2 - 2z + 5$.

(1)

By long division we have

$$z^4 - 5z^3 + 18z^2 - 29z + 35 = (z^2 - 2z + 5)(z^2 - 3z + 7)$$

(1)

and if $z^2 - 3z + 7 = 0$ then

$$z = \frac{3 \pm \sqrt{9 - 28}}{2} = \frac{1}{2}[3 \pm i\sqrt{19}].$$

Thus solutions of $p(z) = 0$ are $z = 1 \pm 2i$ and $z = [3 \pm i\sqrt{19}]/2$.

(1)

29. (a) Sketch the region R of the complex plane defined by

$$|z-1| \geq |z-i| \qquad \text{and} \qquad |z-2(1+i)| \leq 1.$$

(4 marks)

Solution

The line L given by $|z-1| \geq |z-i|$ is the perpendicular bisector of the line joining 1 and i; it is easy to see that this is the line $y = x$ in the complex plane.

(1)

The region $|z-1| \geq |z-i|$ therefore is the region above this line.

(1)

The region $|z-2(1+i)| \leq 1$ is a circle centre $2(1+i)$ and radius 1. Note that the line L passes through the centre C of the circle.

(1)

Hence the required region R is the half of the circle centre $2(1+i)$ and radius 1 that lies above the diameter $x = y$.

(1)

(b) Determine the modulus of the complex number $z \in R$ for which $\arg(z)$ is greatest.

(3 marks)

Solution

Suppose the point Q represents the complex number α with the greatest argument.

Then OQ is a tangent to the circle; then OCQ is a right-angled triangle with right-angle at Q.

(2)

Now $|OC| = \sqrt{8}$ and $|QC| = 1$ so, by Pythagoras, $|OQ| = \sqrt{7}$.

Hence the required complex number has modulus $\sqrt{7}$.

(1)

30. (a) Verify that

$$\left(z - e^{i\varphi}\right)\left(z - e^{-i\varphi}\right) \equiv z^2 - 2z\cos\varphi + 1\,.$$

(1 mark)

Solution

Now

$$\left(z - e^{i\varphi}\right)\left(z - e^{-i\varphi}\right) = z^2 - (e^{i\varphi} + e^{-i\varphi})z + 1 = z^2 - 2z\cos\varphi + 1$$

(1)

(b) Solve $z^5 + 1 = 0$. Give your answers in polar form $re^{i\varphi}$ with $-\pi < \varphi \le \pi$.

(4 marks)

Solution

Since $z^5 = -1 = e^{i\pi}$ then deMoivre's theorem tells us that

(1)

$$z = \left[\cos\left((2k+1)\pi\right) + i\sin\left((2k+1)\pi\right)\right]^{1/5} = \left[\cos\left(\frac{(2k+1)\pi}{5}\right) + i\sin\left(\frac{(2k+1)\pi}{5}\right)\right]$$

(2)

For roots in the specified range we choose $k = -2, -1, 0, 1, 2$ so that $z = \exp(i\varphi)$ with $\varphi = -3\pi/5, -\pi/5, \pi/5, 3\pi/5$ and π.

(1)

(c) **Given** that

$$1 - z + z^2 - z^3 + z^4 = \frac{z^5+1}{z+1}$$

express

$$z^4 - z^3 + z^2 - z + 1$$

as the product of two quadratic factors with real coefficients.

(3 marks)

Solution

Using part (a) with angles $\pi/5$ and $3\pi/5$ we see from part (b) that

$$z^5 + 1 = (z+1)\left(z^2 - 2z\cos\left(\frac{\pi}{5}\right) + 1\right)\left(z^2 - 2z\cos\left(\frac{3\pi}{5}\right) + 1\right) \qquad \text{so}$$

(2)

$$z^4 - z^3 + z^2 - z + 1 = \left(z^2 - 2z\cos\left(\frac{\pi}{5}\right) + 1\right)\left(z^2 - 2z\cos\left(\frac{3\pi}{5}\right) + 1\right)\,. \qquad (1)$$