2022 ASSESSMENT REPORT

MTS415118 MATHEMATICS SPECIALISED

Generally, the mathematical communication demonstrated in student responses was not of the high standard expected by students of this course. For example, $f'' = +$ is not acceptable (this was seen often enough to require a comment). Other sections had a similar standard of examples of unacceptable communication. Students are strongly reminded of the instruction …. "You must show the method used to solve a question. If you only show your answers, you will get few if any marks".

When drawing graphs, too many students did not label axes, did not have a scale on either axis, and did not label important/significant points. The intention of any graph in mathematics is to convey a visual understanding of the relationship between the variables concerned.

SECTION A – SEQUENCES AND SERIES

Overall, this section was quite well done, with a higher average mark across the board than in previous years for this section. The first two questions surprised many students, but the standard nature of Q3 and the other part (a)s, were questions students were well prepared for.

QUESTION 1

This question proved unexpectedly challenging for students, with only a quarter of students achieving full marks. Many students only gave one inequality for an answer, many more gave answers with the inequality the wrong way. Some students gave the answer $-1 > x > 4$ and were penalised ½ a mark for poor notation.

QUESTION 2

More than half of students correctly determined that $r = \frac{1}{2}$ and so $S_\infty = 24$. However, communication was quite poor with this question and often this is all the students wrote. Students received full marks for these two equations.

QUESTION 3

This question was a pleasure to mark. ¾ of students achieved 6 marks or greater in this question. The major fault with students' answers was in not stating how they had used the induction assumption when evaluating the LHS of P_{k+1}. Writing "using assumption", "using equation (*)" or underlining the first *k* terms of P_{k+1} were all accepted.

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QUESTION 4

(a) This question was also well handled. ¾ of students achieved 4 marks or greater. The most common errors were writing that the *kth* term was $V_k - V_{k+1}$ instead of $V_k - V_{k+2}$ or stating that $S_n = V_1 - V_{n+1}$ instead of $S_n = V_1 + V_2 - V_{n+1} - V_{n+2}$. Note that the "V" notation was not used by some students, and this was completely acceptable.

(b) Almost all students did not know how to answer this question, with only 8% achieving full marks. One mark was given for recognising that $S_\infty = \frac{3}{4}$ or writing that $\sum_{k=1}^{N-1} \frac{1}{k(k+2)} + \sum_{k=N}^{\infty} \frac{1}{k(k+2)} = \sum_{k=1}^{\infty} \frac{1}{k(k+2)}$. Most students tried to solve $\frac{1}{k(k+2)} < 0.1$ or $\frac{3}{4} - \frac{1}{n+1} - \frac{1}{n+2} < 0.1$. If they solved these inequalities to an incorrect value of N they received one mark.

QUESTION 5

(a) Most students found this convergence proof difficult. Many made mistakes in putting the LHS over a common denominator, or did not attempt this at all. Most students claimed that $|2(-1)^n - 3| \leq |2 - 3| = |-1| = 1$, rather than $|2(-1)^n - 3| \leq 5$. A number of students knew they needed to rig the numerator, but chose to solve for *n* with complicated algebra rather than rigging with $6n^2 + 4 > n$ in the denominator.

(b) The correct terminology was required here for full marks. Many students said that the sequence converged to $\frac{1}{2}$ and $-\frac{1}{2}$ rather than saying it oscillated finitely and so was a divergent sequence.

QUESTION 6

(a) This question was quite well done, though many students considered the constant 0 a term and so only expanded up to the x^2 term. These students were only penalised one mark. The question stating "give the first three non-zero terms" would have avoided this confusion.

(b) A variety of techniques were employed for this question, most unsuccessfully. Some students ignored 6(a) and started again with the new function; many wrote $1 - 2x + x^2$ as $(x - 1)^2$ which made using 6(a) difficult; others noticed that $\ln(1 - 2x + x^2) = 2\ln(1 - x)$ and were successful. A few students noticed that they could substitute $-2x - x^2$ for x but then refused to expand their answer to give just three terms. Almost all students were well prepared for answering a MacLaurin Series question, it was algebraic errors that caused issues.

SECTION B – MATRICES AND LINEAR TRANSFORMATIONS

Overall students did okay on routine tasks but struggled on harder parts. There was a distinct lack of explanation of what students were doing and many scripts were just a stream of mathematical statements.

QUESTION 7

Well done in almost all cases: most errors occurred through the multiple minus signs when working out -2**A**.

QUESTION 8

Again, most got this one out. Common errors included pre-multiplying B by the inverse of A. Several students could not work out the inverse of A properly.

QUESTION 9

Reasonably well done with the expected range of mistakes. Some did not multiply the two transformation matrices together but assumed the action of the transformations happened simultaneously rather than sequentially. Some got the matrices in the wrong order. A common error was not to invert the transformation so that the student found the pre-image that led to the circle rather than its image. Most knew how to determine the area of the image but were let down by being unable to calculate the determinant correctly. A rather large proportion of the cohort also seem not to know the area of a circle of radius 1.

QUESTION 10

This was more poorly answered than expected; it was quite straightforward for problems of this type. It also revealed a number of apparent misconceptions that should have been addressed before the external assessment stage.

Most students could perform a few row reductions, although the expected number of algebraic slips occurred. Some students went straight for the calculator and just quoted the reduced form; little credit is given for doing this. Many students got the third equation the form

$$(a+3)z=(b+6) \qquad (*)$$

(or something equivalent). This is where the difficulties started. Many divided both sides by (a+3), which is dangerous if one remembers this might be zero.

But the interpretations of (*) in terms of infinitely many/no solutions was concerning. Many students appeared to know something had to be zero, but didn't know what. A common statement was that if a is not -3 and b is not -6 then the equation has infinitely many solutions.

Another common statement was that if b=-6 there are no solutions at all. It is advised that teachers engage with their students to ensure they understand these fundamental concepts.

Most students could do something with part (b) with the specific values of the parameters. Unfortunately, many didn't avail themselves of the reduced form they had already derived in part (a) but worked it all through from scratch.

QUESTION 11

Part a) was very well done. Many of the expected traps were avoided; the matrices were multiplied in the correct order; most of the few errors arose from mistakes in the various trig values.

The quality of the sketches in b) were very variable; there was some leniency given but many of the diagrams were so poor and unclear that some deduction had to be applied.

About half of the students attempted part c) and many of these appeared to see what was going on.

QUESTION 12

This was poorly answered; it probably wasn't helped by part a) not being as clear as it could have been, and many students didn't realise that the rest of the question wasn't dependent on a).

Few knew a good way to tackle a).

Just about everyone who tried b) did so successfully.

Most also knew the important elements required to answer c)

Part d) was rather poorly answered given that testing whether a line is embedded in a plane is explicitly mentioned in the syllabus. Most knew to substitute the equation of the line into that of the plane but didn't know what to do next.

There was evidence of time pressures coming into play by the stage students were tackling Q12.

SECTION C – DIFFERENTIAL CALCULUS, AREAS AND VOLUMES

QUESTION 13

This question was answered very well. Most students received full marks. The few errors tended to be missing negative signs or missing the chain rule.

QUESTION 14

Very few students received full marks for this question. Students who were successful demonstrated good setting out – finding the first and second derivatives separately then working the LHS to get to the RHS. Students are reminded to not assume the result (often this caused confusion, as well as being poor mathematical communication).

QUESTION 15

Generally, this question was not done well.

(a) A reasonable number of students were successful showing the equation of the tangent as given, however, many seemed to be confused by the use of pronumerals (instead of numbers) for the coordinates of point *P*.

(b) Many students assumed that p and q were the gradients of the two tangents and stated $p \times q = -1$ —which was the correct answer from an incorrect assumption—this was awarded ½ mark.

(c) This part was most often not attempted at all. There was a small number of students who were successful. Most students seemed to forget that to find an intersection point they needed to set the two tangent equations equal and solve for x, using $pq = -1$ from part (b) to get to the required result.

QUESTION 16

Finding the first and second derivatives was far simpler if the original division was done

$$i.e.\ \frac{x^2 - 5x + 7}{x - 2} = x - 3 + \frac{1}{x - 2}$$

If this division wasn't done, then many students appeared to struggle with using the quotient rule.

Recognising that $x \neq 2$, and therefore $x = 2$ was the vertical asymptote, was rare.

Quite a few students forgot to find the y-intercept

(a) Most students were able to state there were no zeros and attempted to find the stationary points. Many forgot to show there were no points of inflection.

(b) For full marks, students were expected to show the correct graph shape, all intercepts, vertical asymptote, and stationary points. It was not expected that the oblique asymptote was shown (or even known to exist!).

QUESTION 17

(a) This question was done reasonably well. Errors arose when students tried to rearrange to make y the subject and then only drew the "positive" parts of the curves. Quite a few forgot to show where the region R was. Many forgot to label intersection points and intercepts.

(b) Generally, well done. Common mistakes came from using the incorrect boundaries. Follow through marks were awarded from errors in part (a), as long as these errors didn't make the question easier!

QUESTION 18

Generally, this question was very well done. A few forgot π and others forgot to square each function separately.

SECTION D – TECHNIQUES OF INTEGRATION

In this section, it is particularly easy to obtain/check answers using calculators. Students needed to clearly indicate that they knew how to solve the problem algebraically. Marks were deducted if excessive calculator use was evident, without appropriate working.

QUESTION 19

(a) Mostly well done, but marks were deducted for excessive calculator use. The most common error was forgetting to divide by $\sqrt{3}$. Students should be advised that if doing this question algebraically, the obvious integral obtained would be $\frac{\arctan(\sqrt{3}x)}{\sqrt{3}}$. A giveaway that the calculator was used to obtain the integral was the obtained answer $\frac{\sqrt{3}\arctan(\sqrt{3}x)}{3}$.

(b) Mostly well done, but this question was surprisingly poorly executed by a large number of students. Some were unable to identify that Partial Fractions were required, and consequently used their calculators to obtain the integral and then performed the limit substitutions by hand. This was awarded 1 out 4 marks. A common error was $\int \frac{1}{2-x} dx = \ln|2-x|$, forgetting to divide through by -1.

Absolute values were too commonly forgotten, which did not allow for correct numerical substitution.

QUESTION 20

Most students were aware of the correct trigonometric identity to employ here, but a small number were unable to use it correctly. Some did not identify that $\frac{1}{2\cos^2 2\theta}$ was able to be integrated and tried to further adjust the expression. A common error again was forgetting to divide through by 2 when integrating $\sec^2 2\theta$.

QUESTION 21

This was one of the more successful questions in the section. For some students, this was where they gained half their marks. As would be expected, there were places where students got lost in the algebra (minor errors with constants or negative signs), but it rarely cost them significant marks.

0.5 marks were deducted for forgetting the "+ constant" at the end of the integral.

A significant proportion of responses displayed very poor communication. Students would be advised to lay out their solutions more clearly.

QUESTION 22

Generally, well done. Nearly all students could start this DE correctly.

A surprising number of students seemingly didn't know how to integrate xe^{x^2}. Calculator use was common for this step, and marks were deducted accordingly.

QUESTION 23

A relatively friendly Type 4 homogenous DE, but the algebra involved in substituting y= Vx (or similar) and then simplifying to the point of integration proved challenging for many students. In some cases, errors made the resulting DE easy to solve, and numerous marks were deducted here. In some cases, the question was made harder due to errors, and credit was given to students who made strong attempts at completion.

Students who completed the integration step were generally quite successful at expressing the solution explicitly (y in terms of x). 1 mark was deducted if this step was not completed.

A common error was taking an incorrect 'reciprocal of both sides of the equation', where…

$$\frac{dV}{dx}.x = \frac{2}{3V^2} - \frac{2V}{3}$$

Incorrectly became…

$$\frac{dx}{dV}.\frac{1}{x} = \frac{3V^2}{2} - \frac{3}{2V}$$

This error indicates less than ideal algebraic understanding, and resulted in numerous marks being deducted, since the resulting algebra was quite easy, and the solution is incredibly difficult to express explicitly.

QUESTION 24

Students experienced, by far, the least amount of success with this question.

Most students who attempted this question solved the DE quite easily. If they then substituted the 3 points given, and displayed 3 equations with 3 unknowns, part marks were awarded. Only one more mark was awarded if the reminder of the question was solved correctly using the calculator.

Marks were awarded for solid algebraic attempts at solving for one of the unknown variables.

Only 4 students in the cohort successfully solved for a variable without a calculator. Of these, only 2 students fully solved the problem and were awarded full marks.

SECTION E – COMPLEX NUMBERS

In general, it would appear that students did not allocate sufficient time to this section.

The graphs in both Questions 27 b) and 29 were not done well. Whilst more details will be listed below, too many students did not label axes, did not have a scale on either axis, and did not label important/significant points. The intention of any graph in mathematics is convey a visual understanding of the relationship between the variables concerned.

QUESTION 25

This was quite well done by most students.

QUESTION 26

Part a) was quite well done.

Most students found part b) to be quite challenging, with the result that many did not attempt or did not complete this question. A good number had a reasonable idea of what to do but found the interchange between the indices and the arguments to be difficult to reconcile.

QUESTION 27

Many students scored well in part a). Most were able to solve to find the solutions, however, a large number of responses did not reduce the modulus to $\sqrt{2}$.

Not as many students attempted part b). Following on from the earlier comments, the vast majority of students did not show the distance of the points from the Origin. Angular displacements/arguments were also missing from the diagram. Some students converted their answers into rectangular co-ordinates and were rewarded for this.

QUESTION 28

Most students showed that they had an appropriate method for part a). A number made numerical errors which made part b) either quite difficult and/or quite long.

In part b) some leeway was given for those who had made errors in part a). A number of students had results which did not follow from their answers in part a). A large number knew how to use the Complex Conjugate Root Theorem but did not state the ***necessary*** conditions for it to be employed, namely, that

$z = x + iy$ is a root ***and*** that the co-efficients are real.

QUESTION 29

Many students found this question very difficult. A number did not attempt this question or did not get very far into it.

A number knew that the first inequality was that of an annulus but either did not successfully locate the centre or did not locate/indicate the intercepts. Poorly drawn diagrams did not assist in this regard.

The second inequality was beyond the vast majority of students, particularly at the stage where the inequality signs are removed. As a consequence, the shaded area was on the other side of the boundary condition. Those who did successfully manage to remove the inequality found that the rest of the question flowed nicely.

As stated earlier, it is expected that students label intercepts, centres of circles and points of intersection. Some reasoning should be given as part of any solution. A key indicating the shaded region must be included.

Future students should, as part of their preparation, practice questions which involve removing inequality signs as this skill is also used in the Sequences and Series section of the paper.

QUESTION 30

Unfortunately, too many students did not have sufficient time for this question. Those who did had been well prepared, and whilst there was the usual range of errors, the vast majority knew what they needed to do.

A common error was to assume that the sum from zero to six in part a) meant that there six terms and not seven.

Solutions provided on following page

SECTION A

① $|3-2x|>5 \Rightarrow 3-2x>5$ or $3-2x<-5$

$\Rightarrow 2x<-2$ or $2x>8$

$\Rightarrow x<-1$ or $x>4$

② Let first three terms be $a,\ ar,\ ar^2$.

Then $a(1+r^2)=15$ & $ar=6$

Hence $\dfrac{1+r^2}{r}=\dfrac{5}{2} \Rightarrow 2r^2-5r+2=0$

$\Rightarrow (2r-1)(r-2)=0$

$\Rightarrow r=\tfrac{1}{2}$ or 2

As $ar<a$ must have $r<1$ so $r=\tfrac{1}{2}$.

Then $a=12$ & sum to ∞ is $\dfrac{a}{1-r}=\dfrac{12}{\frac{1}{2}}=24$

③ $$\frac{1}{1.2.3}+\frac{1}{2.3.4}\ldots\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

When $n=1$, L.H.S. $=\tfrac{1}{6}$ & RHS $=\dfrac{1.4}{4.(2)(3)}=\tfrac{1}{6}$

Assume true for $n=N$. Need to prove true for $n=N+1$.

Then $$\sum_1^{N+1}\frac{1}{r(r+1)(r+2)}=\sum_1^{N}\frac{1}{r(r+1)(r+2)}+\frac{1}{(N+1)(N+2)(N+3)}$$

$$=\frac{N(N+3)}{4(N+1)(N+2)}+\frac{1}{(N+1)(N+2)(N+3)}$$ (assuming result true for $n=N$)

$$=\frac{1}{(N+1)(N+2)}\left[\frac{N(N+3)}{4}+\frac{1}{(N+3)}\right]$$

$$=\frac{1}{(N+1)(N+2)}\left[\frac{N(N^2+6N+9)+4}{4(N+3)}\right]$$

$$=\frac{1}{4(N+1)(N+2)(N+3)}\left[N^3+6N^2+9N+4\right]$$

$$=\frac{1}{4(N+1)(N+2)(N+3)}(N+1)(N^2+5N+4)=\frac{(N+1)(N+4)}{4(N+2)(N+3)}$$

as required.

Hence true for all integers n.

(4)

$$\frac{1}{k(k+2)} = \frac{1}{2}\left(\frac{1}{k} - \frac{1}{k+2}\right)$$

Then

$$\sum_{1}^{n} \frac{1}{k(k+2)} = \frac{1}{2}\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right)$$
$$+ \frac{1}{2}\left(\frac{1}{2} - \cancel{\frac{1}{4}}\right)$$
$$+ \frac{1}{2}\left(\cancel{\frac{1}{3}} - \frac{1}{5}\right)$$
$$\vdots$$
$$+ \frac{1}{2}\left(\frac{1}{n-2} - \cancel{\frac{1}{n}}\right)$$
$$+ \frac{1}{2}\left(\cancel{\frac{1}{n-1}} - \frac{1}{n+1}\right)$$
$$+ \frac{1}{2}\left(\cancel{\frac{1}{n}} - \frac{1}{n+2}\right)$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$$

$$= \frac{1}{2}\left[\frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}\right]$$

$$= \frac{1}{2}\left[\frac{3(n^2+3n+2) - 4n - 6}{2(n+1)(n+2)}\right]$$

$$= \frac{3n^2+5n}{4(n+1)(n+2)} = \frac{n(3n+5)}{4(n+1)(n+2)}$$

As $n \to \infty$ so sum to infinity approaches $3/4$

Then

$$\sum_{1}^{N-1} \frac{1}{k(k+2)} + \sum_{N}^{\infty} \frac{1}{k(k-2)} = 3/4$$

$$\therefore \sum_{N}^{\infty} \frac{1}{k(k-2)} = \frac{3}{4} - \frac{(N-1)(3N+2)}{4N(N+1)}$$

$$= \frac{3N(N+1) - (3N+2)(N-1)}{4N(N+1)}$$

$$= \frac{3N(N+1) - (3N^2 - N - 2)}{4N(N+1)}$$

$$= \frac{4N+2}{4N(N+1)} = \frac{2N+1}{2N(N+1)}$$

If sum $< \frac{1}{10}$, $\quad \frac{2N+1}{2N(N+1)} < \frac{1}{10}$

$$\Rightarrow 2N^2 + 2N > 20N + 10$$
$$N^2 - 9N > 5$$
$$(N-4.5)^2 > 25.25$$
$$\Rightarrow N \geq 10$$

5 a) Need to prove that for every $\varepsilon > 0$, $\exists N$ such that

$$\left| \frac{(-1)^n + n^2}{3 + 2n^2} - \frac{1}{2} \right| < \varepsilon \quad \text{when } n > N$$

$$\Rightarrow \left| \frac{2(-1)^n + 2n^2 - 3 - 2n^2}{2(3 + 2n^2)} \right| < \varepsilon$$

$$\Rightarrow \frac{-2(-1)^n + 3}{2(3+2n^2)} < \varepsilon \quad (*)$$

(L.H.S now has to be >0 so modulus signs unnecessary)

Since $\left| \frac{3 - 2(-1)^n}{2(3+2n^2)} \right| \leq \frac{5}{2(3+2n^2)} < \frac{5}{4n^2}$,

statement $(*)$ is true if $\frac{5}{4n^2} < \varepsilon \Rightarrow n^2 > 5/4\varepsilon$

$\Rightarrow n > \sqrt{5/4\varepsilon} = N$

Thus sequence converges to ½ as required.

b) $$\left\{ \frac{1 + (-1)^n n^2}{3 + 2n^2} \right\} = \left\{ \frac{1/n^2 + (-1)^n}{3/n^2 + 2} \right\}$$

and alternates between $\pm \frac{1}{2}$ as $n \to \infty$

Hence the sequence does not converge.

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6 a) If $y = \ln(1+x)$, $y' = \frac{1}{1+x}$, $y'' = -\frac{1}{(1+x)^2}$, $y''' = \frac{2}{(1+x)^3}$

Since $y(0) = 0$, $y'(0) = 1$, $y''(0) = -1$, $y'''(0) = 2$

Maclaurin series is $\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 \ldots$

b) Then

EITHER $\ln(1 + (-2x+x^2)) = (-2x+x^2) - \frac{1}{2}(-2x+x^2)^2 + \frac{1}{3}(-2x+x^2)^3 \ldots$

$= -2x + x^2 - \frac{1}{2}(4x^2 - 4x^3 \ldots) - \frac{8}{3}x^3 \ldots$

$= -2x - x^2 - \frac{2}{3}x^3 \ldots$

OR $\ln(1-2x+x^2) = \ln(1-x)^2 = 2\ln(1-x)$

$= 2\left[-x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]$

$= -2x - x^2 - \frac{2}{3}x^3 \ldots$

SECTION (B)

(7). $A = \begin{pmatrix} -3 & 11 \\ -2 & 5 \end{pmatrix}$: $A^2 = \begin{pmatrix} -3 & 11 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} -3 & 11 \\ -2 & +5 \end{pmatrix} = \begin{pmatrix} -13 & 22 \\ -4 & 3 \end{pmatrix}$

Then $$A^2 - 2A + 7I = \begin{pmatrix} -13 & 22 \\ -4 & 3 \end{pmatrix} - \begin{pmatrix} -6 & 22 \\ -4 & 10 \end{pmatrix} + \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix} = 0$$

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(8) If $CA = B$, $C = BA^{-1}$.

With $A = \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix}$, $A^{-1} = -\frac{1}{3}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}$

Then $$C = -\frac{1}{3}\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} = -\frac{1}{3}\begin{pmatrix} 0 & 3 \\ 2 & -4 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -2/3 & 4/3 \end{pmatrix}$$

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(9) Dilation of factor 3 $\Rightarrow$ $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{array}{l} x' = x \\ y' = 3y \end{array}$

Shear of factor 2 in x $\Rightarrow$ $\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} \Rightarrow \begin{array}{l} x'' = x' + 2y' \\ y'' = y' \end{array}$

Then if new co-ordinates are (X, Y)

$$\left.\begin{array}{l} X = x' + 2y' = x + 6y \\ Y = y' = 3y \end{array}\right\} \Rightarrow \quad y = Y/3, \quad x = X - 2Y$$

If $x^2 + y^2 = 1 \Rightarrow X^2 - 4XY + 4Y^2 + \frac{Y^2}{9} = 1$

Thus equation of image is $X^2 - 4XY + \frac{37}{9}Y^2 = 1$

As determinants of the transformation matrices are 3 and 1 and our original circle is of area π, area of transformed shape is 3π.

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(10).
$$\left[\begin{array}{ccc|c} 2 & 3 & 2 & 10 \\ 3 & 4 & -1 & 4 \\ 1 & 1 & a & b \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 3 & 2 & 10 \\ 0 & -1 & -8 & -22 \\ 0 & -1 & 2a-2 & 2b-10 \end{array}\right] \begin{array}{l} \\ 2R_2 - 3R_1 \\ 2R_3 - R_1 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c} 2 & 3 & 2 & 10 \\ 0 & -1 & -8 & -22 \\ 0 & 0 & 2a+6 & 2b+12 \end{array}\right]$$

Infinitely many solutions if $a = -3$ & $b = -6$

No solution if $a = -3$ & $b \neq -6$.

b)

If $a = -2$, $b = -5$ last equation is $2z = 2 \Rightarrow z = 1$

Then $y = -8z + 22 = 14$

& $2x = 10 - 2z - 3y = -34 \Rightarrow x = -17$.

$\therefore (x, y, z) = (-17, 14, 1)$

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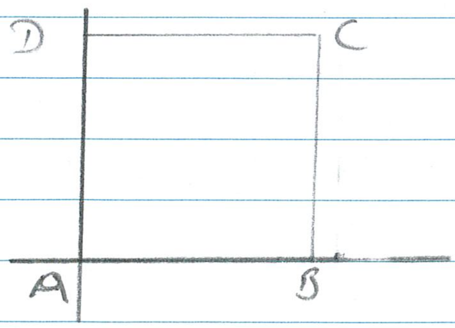
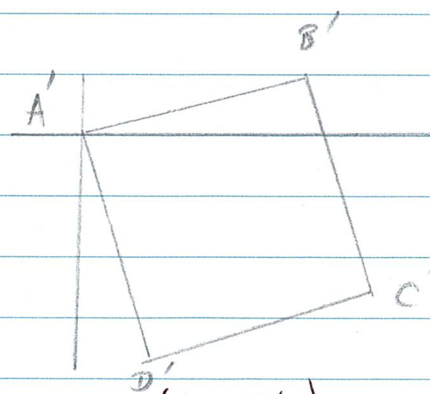
11. The matrix M is
$$\begin{pmatrix} \cos \pi/3 & \sin \pi/3 \\ \sin \pi/3 & -\cos \pi/3 \end{pmatrix} \begin{pmatrix} \cos \pi/6 & -\sin \pi/6 \\ \sin \pi/6 & \cos \pi/6 \end{pmatrix}$$
$$= \frac{1}{4} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \begin{pmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{pmatrix}$$
$$= \frac{1}{4} \begin{pmatrix} 2\sqrt{3} & 2 \\ 2 & -2\sqrt{3} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{pmatrix}$$

Then



$\Rightarrow$



$B' = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$

$C' = \frac{1}{2}\begin{pmatrix} 1+\sqrt{3} \\ 1-\sqrt{3} \end{pmatrix}$

$D' = \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$

M is a reflection in the line $y = x\left(\tan \pi/12\right)$

(12) a) $PQ = (-1, \alpha - 7, 3)$

$PR = (\beta - 2, -3, 9)$

These are parallel (so PQR a straight line) if

$\beta - 2 = 3(-1)$ & $-3 = 3(\alpha - 7)$

$\beta = -1$ & $\alpha = 6$

b) Plane Π is $x + y - 3z = C$.

As contains P, $C = 2 + 7 + 9 = 18$.

c) Parallel plane through Q is $x + y - 3z = K$.

As contains $Q = (1, 6, 0)$, $K = 7$. Hence plane is $x + y - 3z = 7$

d) $x = 1 + t$, $y = -1 + 2t$, $z = \gamma + t\eta$ is line

To lie in plane Π

$$1 + t - 1 + 2t - 3(\gamma + t\eta) = 18$$

$$\Rightarrow -3\gamma + t(3 - 3\eta) = 18.$$

This must be satisfied for all values of t meaning that $\eta = 6$ & $\gamma = -6$

SECTION C

13. a) $\frac{d}{dx}\left[\cos x^2 + \cos^2 x\right] = -2x\sin x^2 - 2\cos x \sin x$

b) $y = 2^{2x} \Rightarrow \ln y = 2x\ln 2$

$\Rightarrow \frac{1}{y} y' = 2\ln 2 \Rightarrow y' = 2y\ln 2 = 2^{2x+1}\ln 2$

c) $y = \arcsin(e^{-x}) \Rightarrow \sin y = e^{-x}$

$\Rightarrow \cos y \; y' = -e^{-x}$

$y' = -\frac{e^{-x}}{\sqrt{1-e^{-2x}}}$

14 $4y^2+1 = 3x^2 \Rightarrow 8yy' = 6x$ (*)

$y' = \frac{3}{4} \cdot \frac{x}{y}$

$\therefore \quad y'' = \frac{3}{4}\left[\frac{y - xy'}{y^2}\right]$

& $y^3 y'' = \frac{3}{4}\left[y^2 - xyy'\right] = \frac{3}{4}\left[y^2 - \frac{3x^2}{4}\right]$ $\left((*) \Rightarrow yy' = \frac{3x}{4}\right)$

$= \frac{3}{4}\left[y^2 - \frac{1}{4}(4y^2+1)\right]$

$= -\frac{3}{16}.$

15. a) $y^2 = 4ax$ so $2yy' = 4a$

At point P, $y' = \frac{4a}{2y} = \frac{4a}{4ap} = \frac{1}{p}.$

Equation of tangent is $y - 2ap = \frac{1}{p}(x - ap^2) \Rightarrow py - 2ap^2 = x - ap^2$

$\Rightarrow py = x + ap^2.$

b) Gradient of tangent at P is $1/p$: gradient tangent at Q is $1/q$.

Perpendicular if $pq = -1$.

c) Now $py = x + ap^2$ & $qy = x + aq^2$ meet when $(p-q)y = a(p^2-q^2)$

$\Rightarrow y = a(p+q)$ and so $x = py - ap^2 = apq$.

As $pq = -1$, $x = -a$ ← tangents meet on this line parallel to y-axis.

16 $y = \frac{x^2 - 5x + 7}{x - 2} = \frac{(x-2)(x-3)+1}{x-2} = x - 3 + \frac{1}{x-2}$

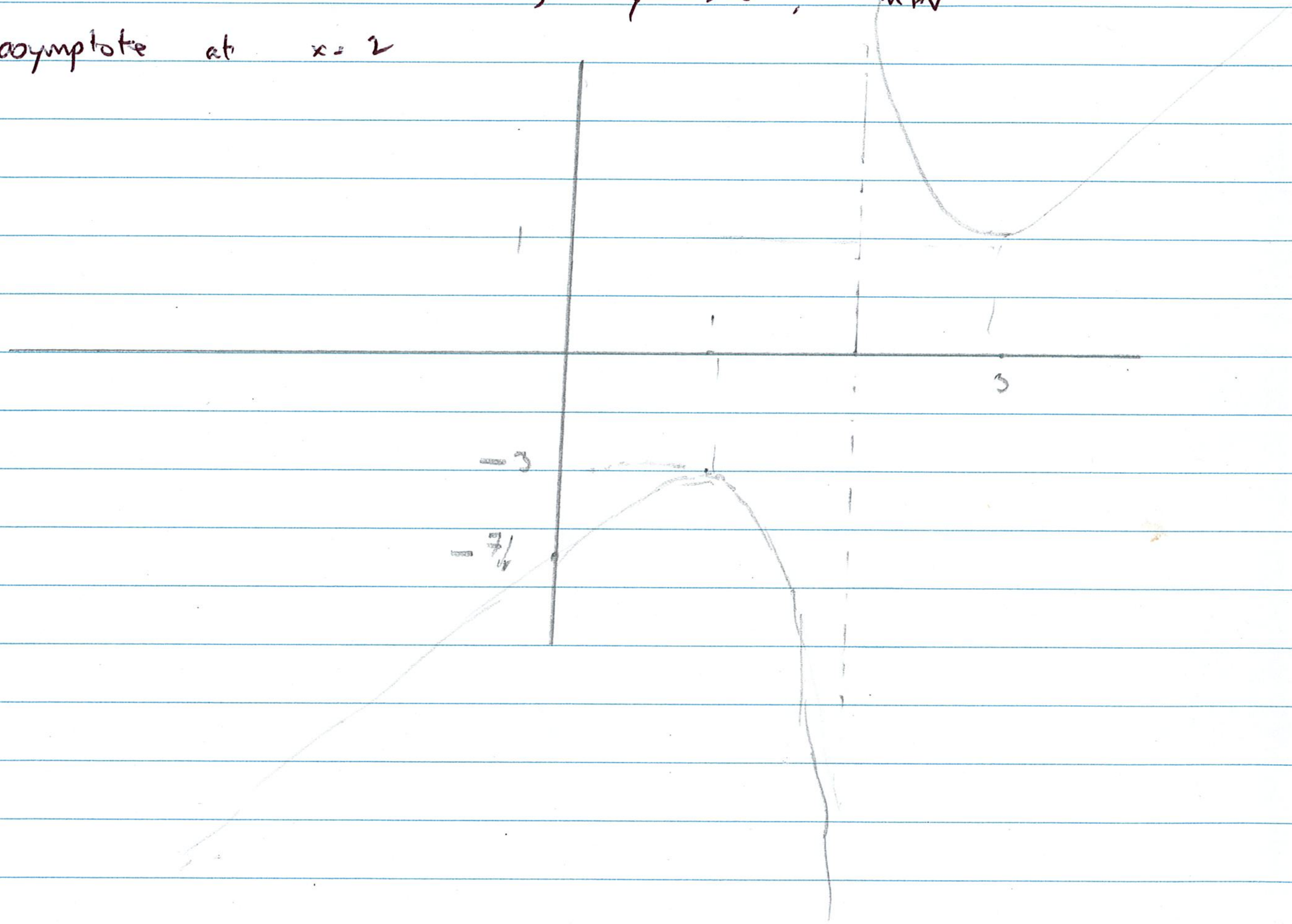
Now $x^2 - 5x + 7 \neq 0$ for all real x. When $x = 0$, $y = -7/2$

$y' = 1 - \frac{1}{(x-2)^2}$ & $y' = 0$ when $x = 1$ or 3

$y'' = \frac{2}{(x-2)^3}$ When $x = 1$, $y'' < 0$, MAX

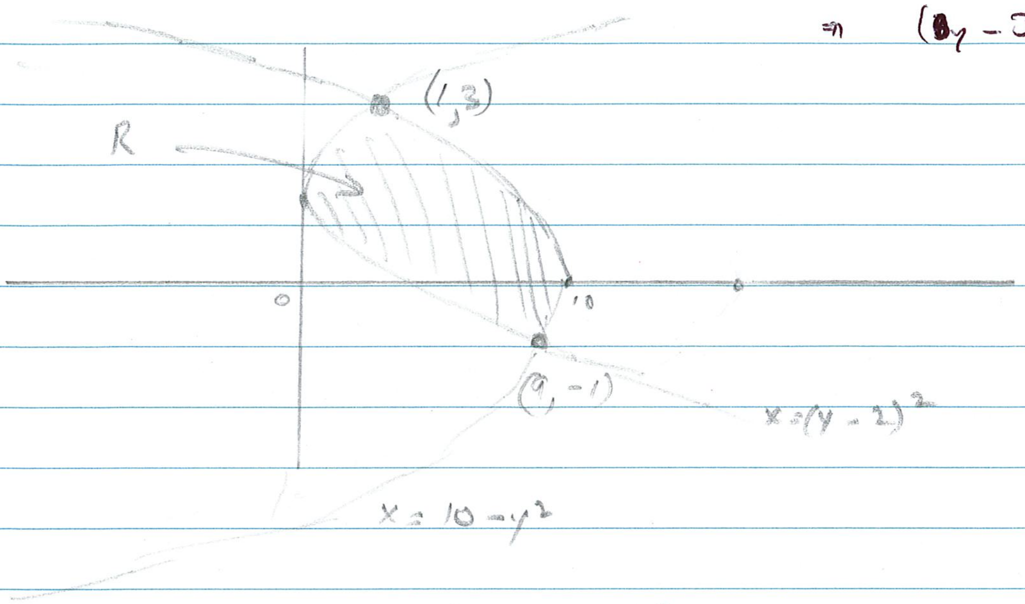
When $x = 3$, $y'' > 0$, min

Vertical asymptote at $x = 2$



17. Curves meet when $10-y^2=(y-2)^2 \Rightarrow 2y^2-4y-6=0$

$\Rightarrow (y-3)(y+1)=0$



Then $R = \int_{-1}^{3} x\,dy = \int_{-1}^{3} \left((10-y^2)-(y-2)^2\right) dy$

$= \int_{-1}^{3} (6+4y-2y^2)\,dy = \left[6y+2y^2-\tfrac{2}{3}y^3\right]_{-1}^{3}$

$= (18+18-18)-(-6+2+\tfrac{2}{3})$

$= 18-(-\tfrac{10}{3}) = \tfrac{64}{3}$ units^2.

18.

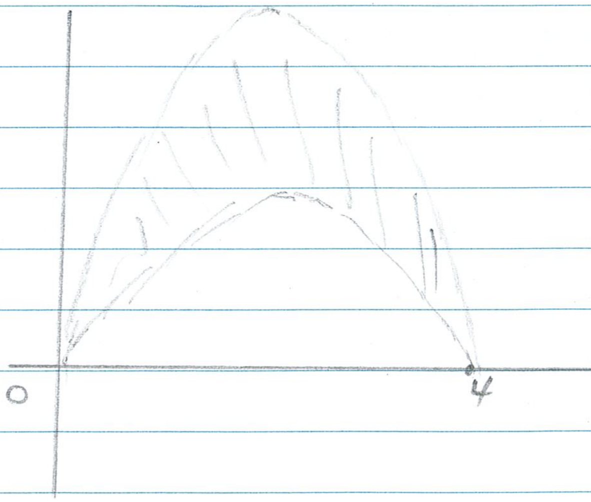
$$V = \pi \int_0^4 [2x(4-x)]^2-[x(4-x)]^2\,dx$$

$= 3\pi \int_0^4 x^2(16-8x+x^2)\,dx$

$= 3\pi\left[\frac{16x^3}{3}-2x^4+\frac{x^5}{5}\right]_0^4$

$= 3\pi.(64).\left[\frac{16}{3}-8+\frac{16}{5}\right] = 3\pi.(512)\left(\frac{2}{3}+\frac{2}{5}-1\right)$

$= 3\pi(512).\left(\frac{10+6-15}{15}\right) = \frac{512\pi}{5}.$



SECTION D

19. a.

$$\int_0^1 \frac{dx}{1+3x^2} = \frac{1}{3}\int_0^1 \frac{dx}{x^2+(1/3)}$$

$$= \frac{1}{3}\left[\sqrt{3}\tan^{-1}(x\sqrt{3})\right]_0^1$$

$$= \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}) = \pi/3\sqrt{3}$$

b)

$$\int_0^1 \frac{dx}{4-x^2} = \frac{1}{4}\int_0^1 \left(\frac{1}{2-x}+\frac{1}{2+x}\right) dx$$

$$= \frac{1}{4}\Big[\ln(2+x) - \ln(2-x)\Big]_0^1$$

$$= \frac{1}{4}(\ln 3)$$

20.

$$\int_0^{\pi/8} \frac{d\theta}{1+\cos 4\theta} = \int_0^{\pi/8} \frac{d\theta}{2\cos^2 2\theta} = \frac{1}{2}\int_0^{\pi/8} \sec^2 2\theta \; d\theta$$

$$= \frac{1}{4}\left[\tan 2\theta\right]_0^{\pi/8} = 1/4$$

21. Let $I = \int e^{-x}\sin 2x \; dx$

$\therefore \; I = -e^{-x}\sin 2x + 2\int e^{-x}\cos 2x \, dx$ (by parts)

$= -e^{-x}\sin 2x + 2\left[-e^{-x}\cos 2x - 2\int e^{-x}\sin 2x \, dx\right]$

$= -e^{-x}(\sin 2x + 2\cos 2x) - 4I$

$\sin 2x$	$-e^{-x}$
$2\cos 2x$	e^{-x}
$\cos 2x$	$-e^{-x}$
$-2\sin 2x$	e^{-x}

$\therefore$

$$5I = -e^{-x}(\sin 2x + 2\cos 2x) + c$$

$$I = -\frac{1}{5}e^{-x}(\sin 2x + 2\cos 2x) + C$$

22. $\dfrac{dy}{dx} = \dfrac{x(e^{x^2}+2)}{6y^2} \Rightarrow 6\int y^2\,dy = \int (x\,e^{x^2} + 2x)\,dx$

$\Rightarrow 2y^3 = \frac{1}{2}e^{x^2} + x^2 + c.$

As $y(0) = 1$, $c = 3/2$

$\Rightarrow \underline{\underline{4y^3 = e^{x^2} + 2x^2 + 3}}$

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23. $\dfrac{dy}{dx} = \dfrac{2x^3+y^3}{3xy^2}$. Put $y = vx$ so $x\dfrac{dv}{dx} + v = \dfrac{2+v^3}{3v^2}$

$x\dfrac{dv}{dx} = \dfrac{2+v^3}{3v^2} - v = \dfrac{2(1-v^3)}{3v^2}.$

Hence $\displaystyle\int \frac{3v^2}{1-v^3}\,dv = 2\int \frac{dx}{x}$

$-\ln(1-v^3) = 2\ln x + \ln C$

$\Rightarrow \ln[(1-v^3)] + \ln x^2 + \ln C = 0$

$= x^2(1-v^3) = K$

When $x = 1$, $y = 2$ so $v = 2 \Rightarrow K = -7$

Then $1 - (y/x)^3 = -7/x^2.$

$\left(\dfrac{y}{x}\right)^3 = \dfrac{x^2+7}{x^2}.$ $\underline{\underline{y^3 = x(x^2+7)}}$

---

(24).

$$\frac{dT}{dt} = -k(T-\Theta)$$

$$\Rightarrow \int \frac{dT}{T-\Theta} = -k\int dt$$

$$\ln(T-\Theta) = \ln C - kt$$

$$T-\Theta = Ce^{-kt}.$$

Given conditions $\Rightarrow$

$$69-\Theta = Ce^{-5k} \quad -(1)$$

$$51-\Theta = Ce^{-10k} \quad -(2)$$

$$39-\Theta = Ce^{-15k} \quad -(3)$$

Now $\frac{(1)}{(2)}$ & $\frac{(2)}{(3)}$ $\Rightarrow$ $e^{-5k} = \frac{51-\Theta}{69-\Theta} = \frac{39-\Theta}{51-\Theta}$

$$\therefore (51-\Theta)^2 = (39-\Theta)(69-\Theta)$$

$$2601 - 102\Theta + \Theta^2 = 2691 - 108\Theta + \Theta^2$$

$$\Rightarrow 6\Theta = 90$$

$$\Theta = 15.$$

Then $e^{-5k} = \frac{51-\Theta}{69-\Theta} = \frac{36}{54} = \frac{2}{3}$ & (1) $\Rightarrow$ $C = 54\left(\frac{3}{2}\right) = 81$

Initial temperature of kettle is $\Theta + C = 96$ degrees.

Temperature of room is $\Theta = 15$ degrees.

SECTION E

25. $$\frac{z}{\sqrt{3}-i} = \frac{1}{\sqrt{3}+i} \Rightarrow z = \frac{\sqrt{3}-i}{\sqrt{3}+i} = \frac{(\sqrt{3}-i)(\sqrt{3}-i)}{(\sqrt{3}+i)(\sqrt{3}-i)} = \frac{2-2\sqrt{3}i}{4}$$

$$= \underline{\underline{\tfrac{1}{2}(1-i\sqrt{3})}}$$

26. a) $$z^2+3z+9=0 \Rightarrow z = \frac{-3\pm\sqrt{-27}}{2} = \frac{-3\pm i\,3\sqrt{3}}{2}$$

$$= -\tfrac{3}{2}(1\pm i\sqrt{3})$$

Now $-1+i\sqrt{3} = 2e^{2i\pi/3}$

so $\underline{\underline{z = 3e^{\pm 2i\pi/3}}}$

If $z_1 = 3e^{2i\pi/3}$, $z_1^N = 3^N\, e^{2i\pi N/3}$

& $z_2 = 3e^{-2i\pi/3}$, $z_2^N = 3^N .\, e^{-2i\pi N/3}$

Equal if $e^{\frac{2i\pi N}{3}} = e^{-\frac{2i\pi N}{3}}$

$e^{\frac{4i\pi N}{3}} = 1$

$\therefore \frac{4N}{3} = 2k \Rightarrow N = \frac{3k}{2}$

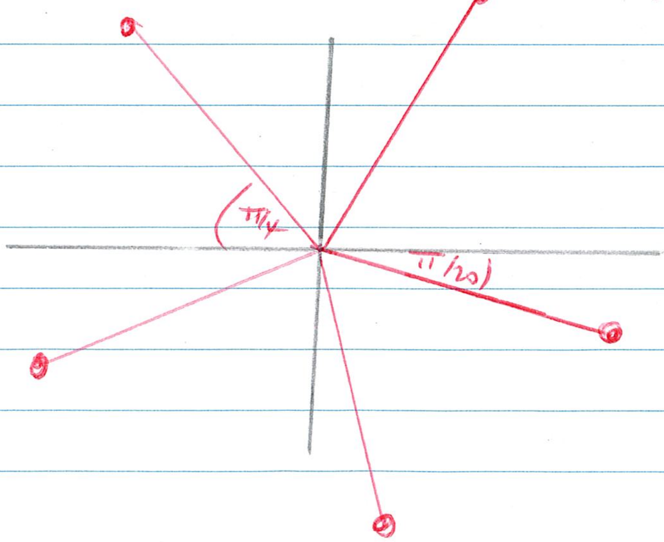
Then N is a multiple of 3

27. If $z^5 = 4(1-i) = 4\sqrt{2}\, e^{-i\pi/4}$

$= 2^{5/2} .\, e^{(2k-1/4)i\pi}$

$$\therefore z = \sqrt{2}\ \exp\left[\frac{(8k-1)i\pi}{20}\right]$$

$z = 2^{1/2}\exp(i\theta)$ with $\theta = -\frac{17}{20}\pi,\ -9\pi/20,\ \pi/20,\ 7\pi/20,\ 3\pi/4$



28 a) $z^4 - 8z^3 + 33z^2 + \alpha z + \beta$

If $z = 2+3i$, $z^2 = -5+12i$, $z^3 = -46+9i$ & $z^4 = -119-120i$

Then $-119 - 120i + 368 - 72i - 165 + 396i + 2\alpha + 3i\alpha + \beta = 0$

Real parts $\Rightarrow$ $84 + 2\alpha + \beta = 0$

Imag. parts $\Rightarrow$ $204 + 3\alpha = 0$ $\underline{\underline{\alpha = -68}}$, $\underline{\underline{\beta = 52}}$

As $z = 2+3i$ is one root, another root is $2-3i$

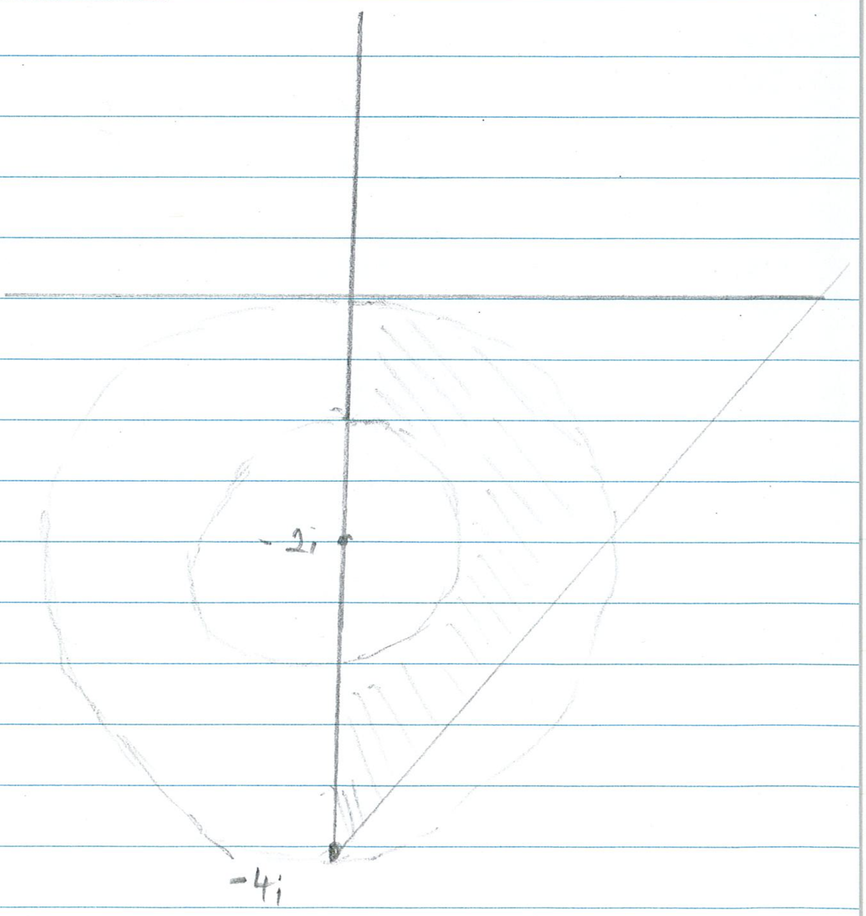
Then $P(z)$ has quadratic factor $(z-(2+3i))(z-(2-3i)) = z^2 - 4z + 13$

By long division $z^4 - 8z^3 + 33z^2 - 68z + 52 = (z^2 - 4z + 13)(z^2 - 4z + 4)$

Hence other roots of $P(z) = 0$ are $z = 2-3i$ & 2 (twice/double root)

29.

$\frac{\pi}{4} \leq \arg(z+4i) \leq \frac{\pi}{2}$



$1 \leq |z+2i| \leq 2$

is annulus centre $-2i$,

inner radius 1, outer radius 2

30 a) $\sum_0^6 z^n = \dfrac{z^7-1}{z-1}$

b) Roots of $z^6+z^5+z^4+z^3+z^2+z+1 = 0$

are solutions of $z^7 = 1$ except $z=1$

If $z^7 = e^{2i\pi k}$, $z = e^{i\theta}$ where $\theta = 0, \pm\frac{\pi}{7}, \pm\frac{2\pi}{7}, \pm\frac{3\pi}{7}$

As excluding $z=1$, roots of polynomial equation are

$z = e^{\pm i\alpha}$, $\alpha = \pi/7, 2\pi/7, 3\pi/7$

c) As $(z - e^{i\alpha})(z - e^{-i\alpha}) = z^2 - 2z\cos\alpha + 1$,
follows that

$$z^6+z^5+z^4+z^3+z^2+z+1 = \left(z^2 - 2z\cos\pi/7 + 1\right)\left(z^2 - 2z\cos 2\pi/7 + 1\right)\left(z^2 - 2z\cos 3\pi/7 + 1\right)$$