2023 ASSESSMENT REPORT

MTS415118 MATHEMATICS SPECIALISED

Generally, the mathematical communication demonstrated in student responses was not of the high standard expected by students of this course. For example, f'' = + is not acceptable (this was seen often enough to require a comment). Other sections had a similar standard of examples of unacceptable communication. Students are strongly reminded of the instruction .... "You must show the methods you used to solve questions to receive full marks".

When drawing graphs, too many students did not label axes, did not have a scale on either axis, or did not label important/significant points. The intention of any graph in mathematics is to convey a visual understanding of the relationship between the variables concerned.

Section A – Sequences and series

Overall, the markers were reasonably impressed with the performance on this section. The markers felt that the majority of candidates appreciated the fundamental ideas behind induction and convergence. While some of the more subtle components of the various arguments were missing, perhaps because for many these things were tackled in the first few minutes of a stressful examination, many candidates did demonstrate competence in this aspect of the course.

Question 1

Most found this question an easy 4 marks. Most of the errors came from sloppy reading and these candidates attempted to write down the 5th and 13th terms of the GP. This led to incorrect equations that couldn't be solved. Others attempted to look at the various sums of the AP and GP and this led nowhere.

Question 2

Just about every candidate knew the basic strategy involved but the explanations were generally poor. Many wrote statements along the lines that we assume that the statement is true for $\boldsymbol{n = k}$ and then played with $\boldsymbol{n = k + 1}$ without really saying the purpose. Just about everyone checked that the statement was true for $\boldsymbol{n = 1}$. The "$\boldsymbol{-6}$" caused problems when adding the $(\boldsymbol{k + 1})$st term as many candidates managed to mix this term up with various powers of 2. Another common mistake was to never record the form of the RHS when we put $\boldsymbol{n = k + 1}$. This had the consequence that when the $(\boldsymbol{k + 1})$st term was added to the assumed sum a series of algebraic manipulations led to something that was confidently stated to be what was needed without ever knowing what that ought to be. Many candidates spoiled their answer by not giving some sort of concluding statement.

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Question 3

With very few exceptions, just about everybody knew the formal statement as to what was required to be proved. Most could battle their way through the manipulations although there were a few mistakes made when dealing with the absolute value. Again, many missed a summarising concluding statement. Overall, though, the impression gained was that most knew the main points in this type of question.

Question 4

Performance was somewhat mixed.

a) Many students said that the cosine oscillates so it must diverge without looking closely whether that is necessarily the case.

b) The answers were better than part (a), with most realising that the argument of exponential became small so that the term tended to 1.

c) Answers to (c) were mixed; about a third knew the result but the rest said that $(1 + (1/n))$ tended to 1 and 1 to any power was 1. A few students thought this tended to ∞.

The most common error in all three parts was to attempt to just substitute "$\boldsymbol{n} = \infty$" and proceed. So, in part (b) the argument of the exponential would be incorrectly written as $\frac{((-\mathbf{1})^{\infty}\infty+\mathbf{1})}{(\infty^{\mathbf{2}}+\mathbf{1})}$, which is of concern.

Question 5

a) Most students were not challenged by this question.

b) Those who knew how to tackle (b) tended to do well although a few got lost in the algebra. It was surprising how many didn't know how to proceed at all.

Question 6

Very easy question although a minority seemed uncertain as to how to construct a Maclaurin Series. There was evidence that many used a calculator to do the differentiation. Several didn't put $\boldsymbol{x} = \mathbf{0}$ in the expressions for the derivates so give a series with coefficients involving peculiar powers of $\mathbf{1} + \boldsymbol{x}$. Surprisingly, there were a number of weaker answers to part (b). Many candidates substituted $\boldsymbol{x} = \mathbf{0.1}$ in their series rather than $\boldsymbol{x} = \mathbf{0.01}$; others put $\boldsymbol{x} = \mathbf{1.01}$ instead. Others only substituted the $\boldsymbol{x}$ value into the linear term in the Maclaurin Series and ignored the rest.

Section B – Matrices and linear transformations

In general, this section was reasonably well done by most students. Most coped well with the standard questions, but often struggled when required to apply their understanding to less familiar contexts.

Question 7

This question was well done by most students. A small number of students interpreted scalars p and q as matrices.

Question 8

This question was also well done by most students. Mistakes were generally made when managing negative signs and fractions.

Question 9

Overall, well done. Typical mistakes here were choosing an anticlockwise rotation instead of clockwise and mistaking the given equation as the image rather than the original. Otherwise, other than calculation errors, this question was well managed.

Question 10

a) Very well handled.
b) Without the angle brackets, many students mistook $(2, \mu, -1)$ as a point. Both interpretations were credited. A large proportion of students did well in this question. A few mistakenly interpreted L to be parallel to the plane given in part a.
c) There were several students that left this part out, possibly due to confusion from part b. Those that attempted it, did well.

Question 11

Students who did not use their calculator to reduce the matrix had much more success than those who did. It was possible to reduce the matrix, without using RREF form, in three or four steps and to proceed from there. This presented a clearer path.

A fair number of students struggled to interpret their results. The most common mistake was defining a unique solution as both $\boldsymbol{\alpha} \neq \mathbf{24}$ **and** $\boldsymbol{\beta} \neq \mathbf{8}$, rather than just $\boldsymbol{\alpha} \neq \mathbf{24}$.

Question 12

a) Very few students managed to solve this question. They struggled to interpret how a line might map onto itself and resorted to trying to substitute a particular point to solve for k or got bogged down with inverse matrices and determinants.
b) This question was better managed than (a). Quite a few successfully found the minimum using the derivative or the turning point of a quadratic. As in part a, those that were successful, were students that had a better visual understanding of the content and were able to make the necessary connection within a slightly different context.

Section C – Differential calculus, areas and volumes

Overall, students performed very well on this section.

Question 13

a) Extremely well done – almost everyone got this correct.

b) Students who simplified the log expression first using log laws were far more successful than those who attempted to apply the Chain Rule. Those who used the Chain Rule rarely managed the algebraic steps without making an error. Advice to students – always simplify log expressions before finding the derivative.

Question 14

Reasonably well done. Difficulties occurred when negative signs were misplaced. Some students used their calculators to find the first and second derivatives, since working was not shown, full marks were not awarded in these cases. Students who were successful demonstrated good setting out – finding the first and second derivatives separately then working the LHS to get to the RHS. Students are reminded to not assume the result (often this caused confusion, as well as being poor mathematical communication).

Question 15

Finding the first and second derivatives was far simpler if the original division was done …

$$\boldsymbol{i.e.\frac{x^2+3}{x-1}=x+1+\frac{4}{x-1}}$$

If this division wasn't done, then some students appeared to struggle with using the quotient rule.

a) Intercepts were found correctly by the majority of students.

   Students need reminding that critical points are points where the derivative is equal to zero (stationary points) or is undefined ($x \neq 1$). A large number of students did not mention that $x \neq 1$ so lost an "easy" mark. In this course, it is expected that stationary points are classified by using the second derivative (sign change of first derivative was acceptable in this case). When using the second derivative, it would be best to mention something about concavity (concave up/down) – just writing f'' = + is not acceptable.

b) For full marks, students were expected to show the correct graph shape, all intercepts, vertical asymptote, and stationary points. It was not expected that the oblique asymptote was shown (or even known to exist).

Question 16

Implicit differentiation was done very well and students showed a good understanding of tangents and normals. Common errors were algebraic ones. Most successful answers came from those who substituted the given point in without first rearranging into $\frac{dy}{dx} = \cdots$ form, and then manipulating the equation to find the gradient of the tangent, etc.

Question 17

Generally, well done. Most common errors came from using the incorrect boundaries. Students are advised to draw the graph to help them to see the required region more easily.

Question 18

Generally, well done. Students are advised to draw the graph to help them see the required region more easily. Quite a few students did not attempt this fairly straightforward volume question.

A significant number of students forgot to include $\boldsymbol{x = 5}$ when determining the region being rotated.

$$\boldsymbol{i.e. Volume = \pi \int_0^4 (5^2 - (5 - 4y + y^2)^2)\, dy}$$

A few forgot to square anything.

Section D – Techniques of integration

Overall, students performed well on this section, with most students displaying a sound knowledge of the various concepts in this section.

Question 19

a) Almost all students achieved full marks on this question, with only a couple failing to use the product rule correctly. A few students tried to integrate $x\ arctanx$ by mistake and received 0 marks.
b) Most students knew how to use part (a) to obtain an expression for $\int arctanx\ dx$, with around half of students gaining full marks for this question. It was required to demonstrate where part (a) was being used (if it was), as well as demonstrate how $\int \frac{x}{1+x^2}\ dx$ could be evaluated.

Question 20

Approximately 45% of students achieved 4 or greater marks here.

Noticing that $\boldsymbol{dx = 2u\, du}$ was key to completing the question and many students only found $\boldsymbol{\frac{du}{dx}}$ and stopped, or did not attempt the question at all. Too many students reached for their calculator to evaluate $\int \boldsymbol{\frac{2u}{1+u}\, du}$ rather than showing it could be represented as $\int \boldsymbol{\left(2 - \frac{2}{1+u}\right) du}$.

Question 21

A number of students did not notice that $\boldsymbol{cosx\ sinx = \frac{1}{2} sin2x}$ and this led to fruitless attempts to integrate the whole of $\boldsymbol{x\ cosx\ sinx}$ by parts. On the whole this question was completed more successfully than the previous one, with the biggest error being keeping track of all of the $\boldsymbol{\frac{1}{2}}$'s along the way.

Question 22

Most students realised this was a separable differential equation and successfully found an expression for the solution. However, to achieve full marks an explicit solution for $\boldsymbol{y}$ in terms of $\boldsymbol{x}$ was needed. Also, many students did not show how $\frac{\mathbf{1}}{\mathbf{1}-\boldsymbol{y}^{\mathbf{2}}}$ could be separated using partial fractions.

Question 23

This question was the most successfully completed question in the whole section, with one third of students achieving full marks and two thirds achieving at least five.

Question 24

a) Approximately 90% of students achieved the one mark for this question.

b) This was the least successfully completed question in the whole section. Two marks were awarded for arriving at the equation $-\ln v = t + k$, with k a constant. Approximately half of all students reached this point, with only a few students going on to achieve full marks.

Common difficulties were:

- not realising that $\boldsymbol{v}$ was a function of $\boldsymbol{C}$, so being faced with integrating $\frac{\mathbf{1}}{\boldsymbol{vC}}$
- not believing the integration constant, $\boldsymbol{k}$ above, was needed as $\boldsymbol{K}$ was already a constant
- using $\boldsymbol{C}$ for the integration constant, then confusing this with the dependent variable
- substituting $\boldsymbol{v = lnK - lnC}$ back into the equation incorrectly, arriving at $\boldsymbol{lnK - lnC = t + k}$ rather than $\boldsymbol{ln(lnK - lnC) = t + k}$.

c) Most students did not attempt this question. Of those that did, many believed that solving $\frac{dC}{dt} = 0$ would find the greatest rate of increase in C, rather than when $\frac{d^2C}{dt^2} = 0$. Less than 5% of students achieved full marks for this question.

Section E – Complex numbers

Completing this section proved to be a challenge for most students, either due to running out of time, or the difficulty of the final question.

Question 25

Most candidates had a lot of success with this question. The most common mistake was not giving the final answer in polar form. A few candidates tried to convert to polar form before simplifying, but this method was rarely successful.

Question 26

Most students combined parts a) and b), by finding $\boldsymbol{vw}$ and $\boldsymbol{v\bar{w}}$ first, and then finding their moduli, rather than recognising the fact that $|\boldsymbol{vw}| = |\boldsymbol{v}||\boldsymbol{w}| = \mathbf{1}$. Most had reasonable success either way, barring a few arithmetic errors.

Most candidates did not make the connection between the numbers and their moduli being 1, and left part c) unanswered.

Question 27

Generally well done. Most candidates knew exactly the method to follow here, even if the levels of execution varied. The most common error was finding the incorrect argument for $\boldsymbol{z^4}$ (correct base angle of $\frac{\pi}{3}$, but incorrect quadrant), but consequential marks were awarded in these cases.

Question 28

Also generally well done. Some candidates combined the two parts, which was usually a quicker method to finish the question. Candidates would be well advised to explain their thinking clearly.

"$\boldsymbol{z = 3i}$ is a root, $\therefore \boldsymbol{P(3i) = 0}$"

"$\boldsymbol{z = 3i}$ is a root and all coefficients are real, $\therefore \boldsymbol{-3i}$ is also a root (conjugate root theorem)"

The above statements should be part of the solution but were lacking in most cases. Marks were generously awarded anyway.

A surprising number of students did not state the final roots, instead giving their answer as a factorised version of $\boldsymbol{P(z)}$.

A large number of students used their calculator to obtain the last 2 solutions from the quadratic factor; 1 mark was deducted in this instance.

Question 29

Most candidates had a solid idea of how to approach part a) of this question. Some students did not square both sides correctly, which made the algebra quite unpleasant. The process of completing the square for both variables was evidently unfamiliar to many students. In other cases, students achieved good success in this question.

Marks for this question were somewhat consequential from part a). Students who drew a circle based on their equation from part a) with everything else required were awarded full marks, provided their error in a) didn't reduce the complexity of part b).

Question 30

This question was an amazing differentiator. Very few students knew how to approach it. Furthermore, it was evident that most did not have time to explore different options.

a) Some were aware this question involved a binomial expansion but did not get very far into it. Some completed the expansion, and equated the imaginary parts correctly, but then did not recognise the substitution using the Pythagorean identity to complete the proof.

b) Similarly, only a handful of students made any headway into this question. A few made the $\theta = \frac{\pi}{5}$ substitution but got no further. Most who recognised that the equation was a quadratic managed to get a reasonable answer (although some used their calculator here). Only one candidate argued that only the negative answer from the quadratic formula should be taken, and this is the only candidate to be awarded 4 marks for this question.

SECTION A

1. AP terms are a, $a+4d$, $a+12d$

GP terms are a, ar, ar^2

If $a = 12$ then $12 + 4d = 12r \Rightarrow 1 + d/3 = r$

& $12 + 12d = 12r^2 \Rightarrow 1 + d = r^2$

Hence $(1 + d/3)^2 = 1 + d \Rightarrow \frac{2d}{3} + \frac{d^2}{9} = d \Rightarrow \frac{1}{9}d^2 = d/3$

$\Rightarrow d = 3 \quad r = 2$

2. $\sum_1^n r^2 . 2^r = 2^{n+1}(n^2 - 2n + 3) - 6$

If $n = 1$, LHS $= 2$, RHS $= 4(2) - 6 = 2$ ✓

Thus OK for $n = 1$.

Assume true for $n = N$,

i.e. $\sum_1^N r^2 . 2^r = 2^{N+1}(N^2 - 2N + 3) - 6$

To prove true for $n = N+1$, i.e. to prove $\sum_1^{N+1} r^2 . 2^r = 2^{N+2}(N^2 + 2N + 1 - 2N - 2 + 3) - 6$

$= 2^{N+2}(N^2 + 2) - 6$

Now $\sum_1^{N+1} r^2 . 2^r = \sum_1^N r^2 . 2^r + 2^{N+1}(N+1)^2 = 2^{N+1}(N^2 - 2N + 3) - 6$

$+ 2^{N+1}(N^2 + 2N + 1)$

$= 2^{N+1}(2N^2 + 4) - 6$

$= 2^{N+2}(N^2 + 2) - 6$

as required.

Hence result proved.

3. If $u_n = \frac{3n+1}{7n-4} \to \frac{3}{7}$ as $n \to \infty$ then $\forall \varepsilon > 0$,

$\exists N$ such that $|u_n - \frac{3}{7}| < \varepsilon$ whenever $n > N$

Now $|u_n - \frac{3}{7}| = \left| \frac{3n+1}{7n-4} - \frac{3}{7} \right| = \left| \frac{21n+7 - 3(7n-4)}{7(7n-4)} \right|$

$= \left| \frac{19}{7(7n-4)} \right| < \frac{19}{7(6n)}$ if $n > 4$

$< \frac{2}{7n}$ if $n > 4$

If $n \geq N = \frac{2}{7\varepsilon}$ then $|u_n - 3/7| < \varepsilon$.

Thus $u_n \to 3/7$ as $n \to \infty$.

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4. a) $\cos\left(\pi(n + 1/n)\right)$. DIVERGES.

As $n \to \infty$ tends to $\cos(n\pi) = (-1)^n$.

b) $\exp\left(\frac{(-1)^n n + 1}{n^2+1}\right)$ CONVERGES to 1

As $n \to \infty$ $\frac{(-1)^n n + 1}{n^2+1} \to 0$

c) $(1 + 1/n)^n$ CONVERGES to e.

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5. $(r+1)^4 = (r^2 + 2r + 1)(r^2 + 2r + 1) = r^4 + 4r^3 + 6r^2 + 4r + 1$.

Then $\left((r+1)^4 - r^4\right) = 4r^3 + 6r^2 + 4r + 1$

$\therefore \sum_1^n \left((r+1)^4 - r^4\right) = 4\left(\sum_1^n r^3\right) + 6\left(\sum_1^n r^2\right) + 4\left(\sum_1^n r\right) + \left(\sum_1^n 1\right)$

$\therefore (n+1)^4 - 1 = 4\left(\sum_1^n r^3\right) + n(n+1)(2n+1) + 2n(n+1) + n$

$= n^4 + 4n^3 + 6n^2 + 4n = 4\left(\sum r^3\right) + n(n+1)(2n+3) + n$

$4\left(\sum r^3\right) = n(n^3 + 4n^2 + 6n + 4 - 1) - n(n+1)(2n+3)$

$= n(n^3 + 4n^2 + 6n + 3) - n(n+1)(2n+3)$

$= n(n+1)(n^2 + 3n + 3) - n(n+1)(2n+3)$

$= n(n+1)(n^2 + n) = n^2(n+1)^2$

$\therefore \sum_1^n r^3 = \frac{1}{4} n^2(n+1)^2$

If $y = (1+x)^{1/2}$, $y' = \frac{1}{2}(1+x)^{-1/2}$, $y'' = -\frac{1}{4}(1+x)^{-3/2}$, $y''' = \frac{3}{8}(1+x)^{-5/2}$

Thus Maclaurin series is

$$y = y(0) + x\,y'(0) + \frac{x^2}{2}y''(0) + \frac{x^3}{6}y'''(0) \ldots$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{1}{16}x^3 \ldots$$

If we put $x = 0.01$ we get

$$\sqrt{1.01} = 1 + \frac{0.01}{2} - \frac{(0.01)^2}{8} + \frac{(0.01)^3}{16} \ldots$$

$$= 1.00498756\ldots$$

SECTION B

7. If $M = \begin{pmatrix} -3 & 2 \\ 4 & 5 \end{pmatrix}$, $M^2 = \begin{pmatrix} -3 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 17 & 4 \\ 8 & 33 \end{pmatrix}$

Then

$$pM + qI = \begin{pmatrix} -3p+q & 2p \\ 4p & 5p+q \end{pmatrix} = \begin{pmatrix} 17 & 4 \\ 8 & 33 \end{pmatrix} \quad \text{iff} \quad \begin{matrix} p = 2 \\ q = 23 \end{matrix}$$

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8. $A = \begin{pmatrix} 2 & c \\ -1 & -1 \end{pmatrix}$ $\quad \det A = c - 2$

$\therefore A^{-1} = \frac{1}{c-2}\begin{pmatrix} -1 & -c \\ 1 & 2 \end{pmatrix}$

If $A + A^{-1} = I$ $\quad \begin{pmatrix} 2 - \frac{1}{c-2} & c - \frac{c}{c-2} \\ -1 + \frac{1}{c-2} & -1 + \frac{2}{c-2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then $\frac{1}{c-2} = 1 \Rightarrow c = 3$. All other elements work

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9. Rotation matrix through angle $\pi/3$ clockwise is $\begin{pmatrix} \cos\pi/3 & \sin\pi/3 \\ -\sin\pi/3 & \cos\pi/3 \end{pmatrix}$

$$R = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$$

Shear of factor $\sqrt{3}$ in y-direction is $\begin{pmatrix} 1 & 0 \\ \sqrt{3} & 1 \end{pmatrix}$

Then the image of (x, y) is $\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} x + \sqrt{3}y \\ y - \sqrt{3}x \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x + \sqrt{3}y \\ 4y \end{pmatrix}$$

Thus $x' = \frac{1}{2}(x + \sqrt{3}y)$, $y' = 2y \Rightarrow y = \frac{1}{2}y'$

& $x = 2x' - \sqrt{3}y = 2x' - \frac{\sqrt{3}}{2}y$.

Then ellipse transformed to

$$2[4x^2 - 2\sqrt{3}xy + \tfrac{3}{4}y^2] + 3(\tfrac{1}{4}y^2) = 1$$

$$8x^2 - 4\sqrt{3}xy + \tfrac{9}{4}y^2 = 1.$$

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10 a) If $(1, -2, 3)$ lies on plane $3 - 8 - 6 = d$. $d = -11$

b) Equation of L is $(x, y, z) = (1, -2, 3) + t[2, p, -1]$

$$= (1 + 2t, -2 + pt, 3 - t)$$

c) If line embedded in plane

$$3(1+2t) + 4(-2+pt) - 2(3-t) = -11$$

$$(6 + 4p + 2)t = 0$$

$$4(p + 2)t = 0 \qquad p = -2$$

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11. $\left[\begin{array}{ccc|c} 1 & 1 & 4 & 1 \\ 3 & 2 & 16 & 5 \\ 4 & 2 & \alpha & \beta \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 4 & 1 \\ 0 & -1 & 4 & 2 \\ 0 & -2 & \alpha-16 & \beta-4 \end{array}\right]$ $R_2 - 3R_1$, $R_3 - 4R_1$

$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 4 & 1 \\ 0 & -1 & 4 & 2 \\ 0 & 0 & \alpha-24 & \beta-8 \end{array}\right]$ $R_3 - 2R_2$

From last line we conclude that

i) if $\alpha \neq 24$, unique solution.

ii) if $\alpha = 24$ & $\beta = 8$ there are infinitely many solutions

iii) if $\alpha = 24$ & $\beta \neq 8$ then we no solution.

When $\alpha = 24$ & $\beta = 8$, $-y + 4z = 2 \Rightarrow y = 4z - 2$

& $x = 1 - 4z - y = 3 - 8z$

$\therefore$ When $\alpha = 24, \beta = 8$, $(x, y) = (3 - 8z, 4z - 2)$

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12 $T = \begin{pmatrix} k & -2 \\ 1-k & k \end{pmatrix}$

If point $\begin{pmatrix} \alpha \\ 2\alpha \end{pmatrix}$ is mapped to $\begin{pmatrix} k & -2 \\ 1-k & k \end{pmatrix}\begin{pmatrix} \alpha \\ 2\alpha \end{pmatrix} = \begin{pmatrix} (k-4)\alpha \\ (1+k)\alpha \end{pmatrix}$

This is also on $y = 2x$ if $1 + k = 2(k-4)$

$k = 9$

Now $\det T = k^2 - 2k + 2$

$= (k-1)^2 + 1$

Now area of image $= |\det T|$ which is least when $k = 1$

$\therefore$ Smallest image when $k = 1$

SECTION C

13. i) $\frac{d}{dx}\left(x^2 e^{\sin x}\right) = 2x e^{\sin x} + x^2 \cos x . e^{\sin x}$

ii) $\frac{d}{dx}\left[\ln\left(e^x . \left(\frac{x-2}{x+2}\right)^{3/4}\right)\right] = \frac{d}{dx}\left[x + \frac{3}{4}\ln\left(\frac{x-2}{x+2}\right)\right]$

$= \frac{d}{dx}\left[x + \frac{3}{4}\ln(x-2) - \frac{3}{4}\ln(x+2)\right]$

$= 1 + \frac{3}{4}\left(\frac{1}{x-2} - \frac{1}{x+2}\right)$

$= 1 + \frac{3}{(x^2-4)} = \frac{x^2-1}{(x^2-4)} = \frac{x^2-1}{x^2-4}$

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14. $y = (\sin^{-1} x)^2$

$\Rightarrow \frac{dy}{dx} = \frac{2\sin^{-1} x}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\,\frac{dy}{dx} = 2\sin^{-1} x$

Diff. again $\Rightarrow \sqrt{1-x^2}.\,\frac{d^2y}{dx^2} - \frac{x}{\sqrt{1-x^2}}\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$

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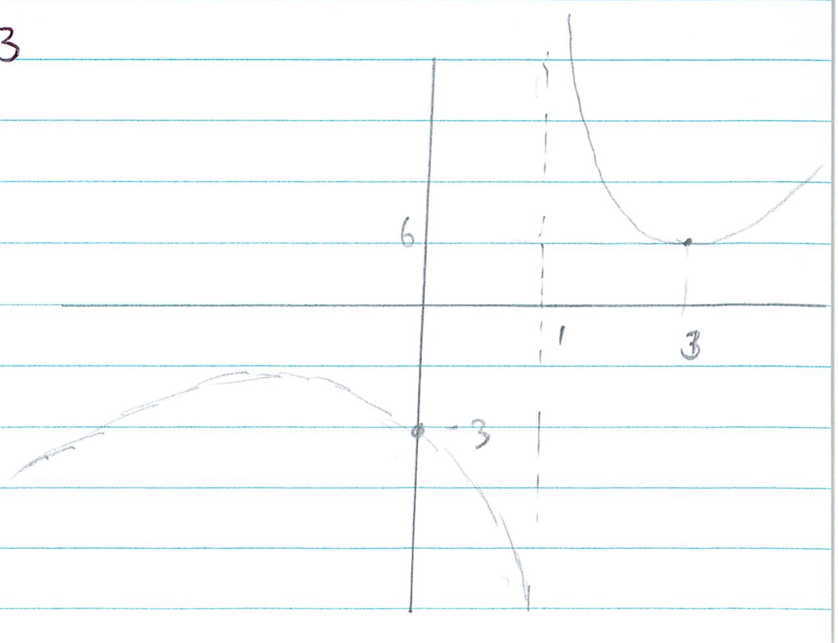
15. $f = \frac{x^2+3}{x-1} \Rightarrow f' = \frac{(x-1).2x - (x^2+3)}{(x-1)^2} = \frac{x^2-2x-3}{(x-1)^2}$

Then $f' = 0$ at $x = -1, 3$

$f(-1) = -2, \quad f(3) = 6$

When $x = 0$, $f = -3$

No values of x for which $f = 0$.



16. $x^2 + 8y^2 = (2x^2 + 2y^2 - x)^2$

Then $2x + 16yy' = 2(2x^2 + 2y^2 - x)[4x + 4yy' - 1]$

When $x = 1, y = 1$: $2 + 16y' = 2(3)(3 + 4y')$

$2 + 16y' = 18 + 24y'$

$8y' = -16$

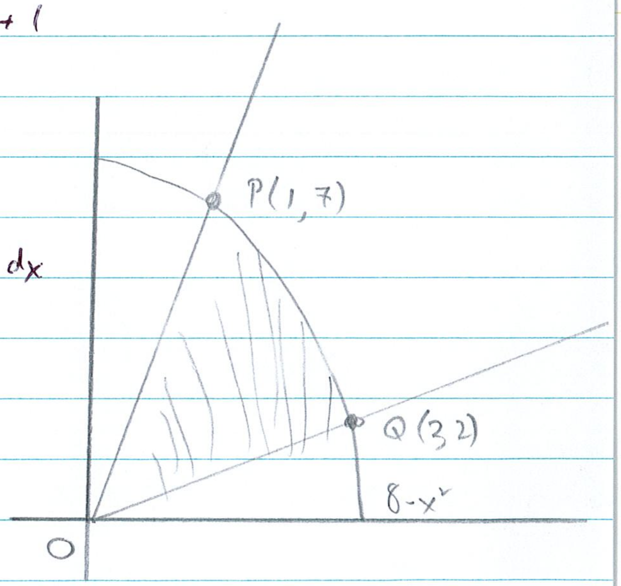
$y' = -2$

$\therefore$ Equation of normal is $y - 1 = \frac{1}{2}(x - 1)$

$2y - 2 = x - 1$

$2y = x + 1$

17.



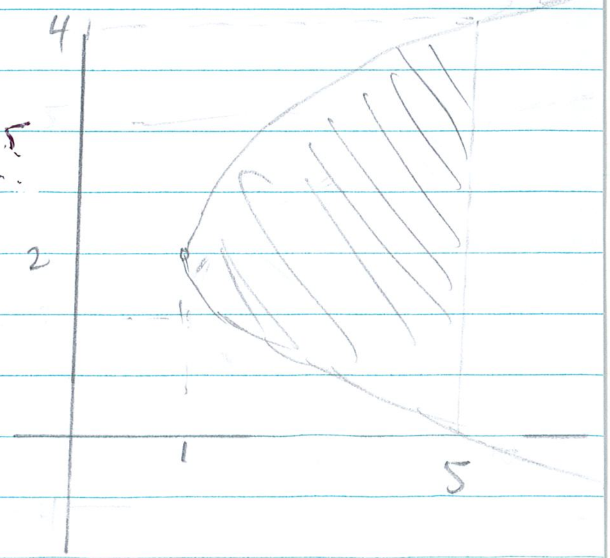
$A = \int_0^1 (7x - 2x)\, dx + \int_1^2 \left((8 - x^2) - 2x\right) dx$

$= \left[\frac{5}{2}x^2\right]_0^1 + \left[8x - \frac{x^3}{3} - x^2\right]_1^2$

$= \frac{5}{2} + 16 - \frac{8}{3} - 4 - 8 + \frac{1}{3} + 1$

$= 5 + \frac{5}{2} - \frac{7}{3} = \frac{30 + 15 - 14}{6} = 31/6$

18



$x = 1 + (y-2)^2$

Area is contained between $x = 5 - 4y + y^2$ & $x = 5$

$V = \pi \int_0^4 \left[5^2 - (5 - 4y + y^2)^2\right] dy$

$= \pi \int_0^4 25 - (25 - 40y + 26y^2 - 8y^3 + y^4)\, dy$

$\pi \int_0^4 (40y - 26y^2 + 8y^3 - y^4)\, dy$

$\left[20y^2 - \frac{26y^3}{3} + 2y^4 - \frac{y^5}{5}\right]_0^4$

$\frac{88\pi}{15}$

SECTION D

19. $$\frac{d}{dx}\left(x\tan^{-1}x\right) = \tan^{-1}x + \frac{x}{x^2+1}$$

$$\therefore \int_0^1 \tan^{-1}x \, dx = \left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{x^2+1}\,dx$$

$$= \tan^{-1}(1) - \frac{1}{2}\left(\ln(x^2+1)\right)_0^1 = \pi/4 - \frac{1}{2}\ln 2$$

---

20. $u = \sqrt{x}$, $\quad du = \frac{1}{2}(\sqrt{x})^{-1}dx \Rightarrow du = \frac{dx}{2u}$,

Then $$\int_4^9 \frac{dx}{1+\sqrt{x}} = \int_2^3 \frac{2u\,du}{1+u} = 2\int_2^3\left(1 - \frac{1}{1+u}\right)du$$

$$= 2\left[u - \ln(1+u)\right]_2^3$$

$$= 2\left[1 - \ln(4/3)\right]$$

---

21. $$\int_0^{\pi/2} x\sin x\cos x\,dx = \frac{1}{2}\int_0^{\pi/2} x\sin 2x\,dx$$

x	$-\frac{1}{2}\cos 2x$
1	$\sin 2x$

$$= \frac{1}{2}\left[\left[-\frac{x}{2}\cos 2x\right]_0^{\pi/2} + \frac{1}{2}\int_0^{\pi/2}\cos 2x\,dx\right]$$

$$= \frac{1}{2}\left[-\frac{\pi}{4}(-1) + \frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/2}\right] = \pi/8$$

---

22 $\quad (1+x^2)\dfrac{dy}{dx} = (y^2-1) \Rightarrow \displaystyle\int \frac{dy}{y^2-1} = \int \frac{dx}{1+x^2}$

$$\Rightarrow \frac{1}{2}\int \left(\frac{1}{y-1} - \frac{1}{y+1}\right) dy = \tan^{-1}x + \frac{1}{2}C$$

$$\ln|y-1| - \ln|y+1| = C + 2\tan^{-1}x$$

Thus $\dfrac{y-1}{y+1} = K e^{2\tan^{-1}x}$

$$y - 1 = K e^{2\tan^{-1}x}(y+1)$$

$$y\left[1 - K e^{2\tan^{-1}x}\right] = 1 + K e^{2\tan^{-1}x}$$

$$y = \frac{1 + K e^{2\tan^{-1}x}}{1 - K e^{2\tan^{-1}x}}$$

---

23 $\quad xy^2y' = x^3 + y^3$

Put $y = vx$ so $\quad x^3 v^2\left(v + x\dfrac{dv}{dx}\right) = x^3(1+v^3)$

$$\Rightarrow xv^2\frac{dv}{dx} = 1 \Rightarrow \int v^2\,dv = \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{3}v^3 = \ln x + C$$

Now $v = 0$ when $x = e \Rightarrow C = -1$

Thus $\left(\dfrac{y}{x}\right)^3 = 3(\ln x - 1)$ so $y^3 = 3x^3(\ln x - 1)$

---

24 $\quad \dfrac{dC}{dt} = C\ln\left(\dfrac{K}{C}\right) \Rightarrow \displaystyle\int \frac{dC}{C[\ln K - \ln C]} = \int dt$

$v = \ln K - \ln C$ so $\quad \displaystyle\int -\frac{dv}{v} = \int dt$

$dv/dC = -1/C$

$$-\ln v = \ln G + t \Rightarrow v = He^{-t}$$

$\therefore \ln K - \ln C = He^{-t}$. If $C = C_0$ when $t = 0$, $H = \ln(K/C_0)$

Then $\ln C = \ln K - He^{-t} \Rightarrow \ln(C/K) = \ln(C_0/K)\,e^{-t}$

Hence $C = K\exp\left[\ln\left(\dfrac{C_0}{K}\right)e^{-t}\right]$

$$\frac{dC}{dt} = C \ln\left(\frac{K}{C}\right)$$

Need to maximise $C \ln\left(\frac{K}{C}\right)$

$$\frac{d}{dC}\left[C \ln\left(\frac{K}{C}\right)\right] = \frac{d}{dC}\left[C \ln K - C \ln C\right]$$

$$= \ln K - \ln C - 1$$

$$= \ln\left(K/Ce\right)$$

This vanishes when

$$C = K/e.$$

Section E.

25. $\dfrac{1}{(2-i)^2} = \left(\dfrac{1}{2-i}\right)^2 = \dfrac{(2+i)^2}{5^2} = \dfrac{3+4i}{25}$

Then $\dfrac{1}{(2+i)^2} = \dfrac{3-4i}{25}$

$\dfrac{1}{(2+i)^2} - \dfrac{1}{(2-i)^2} = \dfrac{-8i}{25} = \dfrac{8}{25} e^{-i\pi/2}$

---

26 $|z_1 z_2| = 1, \quad |z_1 \bar{z}_2| = 1$

$z_1 z_2 = \dfrac{1}{65}(3+4i)(5+12i) = \dfrac{1}{65}(-33+56i)$

$z_1 \bar{z}_2 = \dfrac{1}{65}(3+4i)(5-12i) = \dfrac{1}{65}(63-16i)$

As both these have modulus 1, $33^2+56^2 = 65^2$
& $63^2+16^2 = 65^2$

---

27 $z^4 = -1-i\sqrt{3} = 2\exp(\tfrac{4}{3}i\pi)$

$= 2\left[\exp\left(\left(\tfrac{4}{3}+2n\right)i\pi\right)\right]$

By de Moivre theorem $z = 2^{1/4}\exp\left(\left(\dfrac{\pi}{3}+\dfrac{n\pi}{2}\right)i\right)$

Putting $n = 0,1,2,3$ gives
$z = 2^{1/4}\exp(i\phi), \quad \phi = \dfrac{\pi}{3}, \dfrac{5\pi}{6}, \dfrac{4\pi}{3}, \dfrac{11\pi}{6}$

Within required range,
$z = 2^{1/4}\exp(i\phi)$
with $\phi = -\dfrac{2\pi}{3}, -\dfrac{\pi}{6}, \dfrac{\pi}{3}, \dfrac{5\pi}{6}$

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28 a) If $z = 3i$ is one root of $P(z) = 0$ then

$$81 + \alpha(-27i) + \beta(-9) + 54i + 45 = 0$$

Real parts $\Rightarrow$ $126 - 9\beta = 0$. $\beta = 14$

Imag. parts $\Rightarrow$ $-27\alpha + 54 = 0$ $\alpha = 2$

Thus $P(z) = z^4 + 2z^3 + 14z^2 + 18z + 45$.

b)

If $z = 3i$ is a root, so is the conjugate $-3i$ so $z^2 + 9$ is a factor.

Now $z^4 + 2z^3 + 14z^2 + 18z + 45 = (z^2 + 9)(z^2 + 2z + 5)$

If $z^2 + 2z + 5 = 0 \Rightarrow z = -1 \pm 2i$

Thus roots of $P(z) = 0$ are $z = \pm 3i, \ -1 \pm 2i$

---

29 $|z + 1| = \sqrt{2}\,|z - i|$

If $z = x + iy \Rightarrow (x+1)^2 + y^2 = 2[x^2 + (y-1)^2]$

$$x^2 + y^2 + 2x + 1 = 2x^2 + 2y^2 - 4y + 2$$

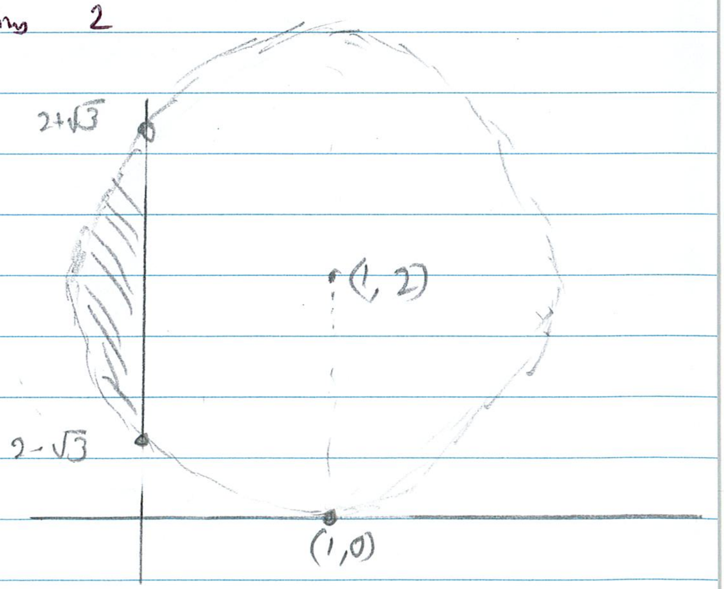
$$x^2 + y^2 - 2x - 4y + 1 = 0$$

$$(x-1)^2 + (y-2)^2 = 4$$

This is a circle, centre $(1, 2)$ radius 2

When $x = 0$, $(y - 2)^2 = 3$

$y = 2 \pm \sqrt{3}$



30. If $z = e^{i\theta}$, $z^5 = e^{5i\theta} = \cos 5\theta + i \sin 5\theta$

Then $\cos 5\theta + i \sin 5\theta = (\cos\theta + i \sin\theta)^5$

$$= \cos^5\theta + 5\cos^4\theta\,(i\sin\theta) + 10\cos^3\theta\,(i\sin\theta)^2 + 10\cos^2\theta\,(i\sin\theta)^3 + 5\cos\theta\,(i\sin\theta)^4 + (i\sin\theta)^5$$

Imaginary parts $\Rightarrow$ $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$$= 5s(1-s^2)^2 - 10s^3(1-s^2) + s^5 \quad \left(\begin{array}{l} s \equiv \sin\theta \\ c \equiv \cos\theta \end{array}\right)$$

$$= 16s^5 - 20s^3 + 5s$$

$\therefore$ $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

If $\theta = \pi/5$, $16s^5 - 20s^3 + 5s = 0$

Either $s = 0$ or $16s^4 - 20s^2 + 5 = 0$

$$s^2 = \frac{20 \pm \sqrt{80}}{32} = \frac{10 \pm 2\sqrt{5}}{16}$$

Thus $s = \pm \frac{1}{4}\sqrt{10 \pm 2\sqrt{5}}$

Now $\sin \pi/5 > 0$ and need to choose $-$ve sign inside square root ($\sin 2\pi/5$ also must satisfy the same equation and is larger than $\sin \pi/5$ so will be associated with the $+$ sign)

$\therefore$ $\sin(\pi/5) = \frac{1}{4}\sqrt{10 - 2\sqrt{5}}$