2024 ASSESSMENT REPORT

MTS415118 MATHEMATICS SPECIALISED

General Comments

The 2024 Mathematics Specialised exam contained a mixture of questions. Some questions were designed to be straightforward but there were also some difficult discriminating questions, some of which were more challenging than usual.

On Sections A and B, the majority of students were able to show an excellent understanding of the concepts covered. In fact, Section B was more straightforward than previous years' papers. Section C Question 14 challenged many students' algebraic skills whilst the remainder of the section was done well. In Section D, the markers acknowledge that Questions 21 and 24 were beyond the ability of many students in an exam setting and that these consumed more than their fair share of time. Section E was achievable if unaffected by time issues.

Written Component

Section A – Criterion 4

Overall, students performed well on this section. Approximately 75% of students scored more than 17 marks. Understanding of the concepts covered was demonstrated; however, sometimes not as well communicated.

Question 1

Generally, satisfactory understanding of absolute value was shown. The greater than or equal to sign took this question away from the more common use of absolute value in this course, which led to some confusion about the direction of the inequality sign when the absolute value signs were removed. The most common error was to state that $-6 \geq 9 - 2x \geq 6$.

Question 2

This question was done well with the majority of students making good progress here. As usual, the difficult point was the algebraic manipulation in the P_{k+1} step. Students are reminded that they are using the assumption that P_k is correct so this must be stated at some relevant point (½ a mark was lost if this connection was not seen). The use of n instead of k (or any letter other than n) at the assumption step was more common than preferred – since we are proving that this statement is true for all n then assuming it is true for n is very poor mathematical communication.

Question 3

Mostly a strong understanding of convergence proofs was demonstrated.

A surprising number of students chose to use the "rigging" method to manipulate the algebraic steps. This method was not required here (no marks were lost if done successfully). Only a few students justified the removal of the absolute value signs (i.e. $n > 4$ or similar). Many students did not write a concluding statement – marks were lost if the final line of proof was left as "$choose\ N = \frac{73+18\varepsilon}{4\varepsilon}$ "– this is not sufficient in a formal proof, some connection to the definition should be stated.

Question 4

a. This question was done very well. Small algebraic errors were the main error here.

b. Again, this was done well. A few students forgot to eliminate the non-zero, non-integer solutions.

Question 5

a. Almost all students were able to set up the difference method correctly with the majority able to get to the $v_1 + v_2 - v_{n+1} - v_{n+2}$ step. A significant number of students made algebraic errors after this step – these were commonly minus sign errors!

b. A good number of students recognised that $\sum_{r=N+1}^{\infty} \frac{1}{r(r+2)} = \sum_{r=1}^{\infty} \frac{1}{r(r+2)} - \sum_{r=1}^{N} \frac{1}{r(r+2)}$

After this point, difficulties with the manipulation of the inequality were seen.

Question 6

a. This question was done very well. Students are reminded to simplify their answer (i.e. no factorials seen in final answer).

b. This question used the word "hence" so some connection to part a) was needed to obtain full marks. This question was done poorly. Many students did not manipulate the log expression at all

$\ln\left(\frac{1}{1-2x}\right) = \ln 1 - \ln(1-2x) = -\ln(1-2x)$ … or similar.

A common substitution attempt was to use $\frac{1}{1-2x} = 1 + \frac{2x}{1-2x}$ and replace x in part a) by $\frac{2x}{1-2x}$. This was awarded full marks if done correctly; however, it was rarely successful due to the more complicated algebraic steps required.

c. The instruction to "Use your series in part b) …" was given, so this needed to happen for full marks to be awarded. If students correctly used part a) instead, then 1 mark was awarded as this choice simplified the question being asked.

Section B – Criterion 5

This section was easier than last year and was completed exceptionally well by most students.

Question 7

a. Almost all students answered this question correctly. There were a variety of solutions to the equation $-x + 10 = -2$, but only $x = 12$ received full marks.

b. Most students calculated the inverse correctly, though some did not make the correct alterations to the matrix A. If students arrived at an incorrect value of x in 7 a), they could still receive full marks here.

Question 8

a. The only issue with this question was with students using poor proof technique, i.e. starting with $M^2 = 7M - 2I$ when this was what needed to be proved. These students lost ½ of a mark.

b. Many students were unsure how to deal with a point involving a variable, with some producing the matrix $\begin{pmatrix} 2k & 8 \\ 2k & -5 \end{pmatrix}$ and not getting any further. Other students did not use the inverse of M in their answer which produced the wrong answer. Using M^{-1} was far more successful than obtaining equations for x and y in terms of $2k + 8$ and $2k - 5$, then solving.

   Overall, only a third of students received full marks for this question.

Question 9

a. This question was quite well done, with around half of students achieving full marks. Common errors were not using the inverse matrix, not calculating the determinant correctly and not expanding the image equation.

b. Two thirds of students received full marks for this question. The most common error was not calculating the area of the circle correctly.

Question 10

a. and b. were both answered extremely well, with most students realising immediately that $c = -6$.

c. A significant number of students tried to show point Q lied on Π_2, even though they had just done this in part b). Otherwise, three quarters of students received full marks.

d. Only a third of students gained full marks for this part. Many made no attempt. The most common error was seeing the point $P(3, 2, 1)$ as a direction vector <3, 2, 1> and using an incorrect method.

   Correct answers given in symmetric form rather than parametric form were given full marks.

Question 11

a. Around two thirds of students gained full marks for this question, which was excellent! The main issue for the rest was choosing incorrect angles to substitute into the transformations. Only a handful of students multiplied the matrices in the incorrect order.

b. Around half of students answered this question correctly. The most common error was interpreting the transformation $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ as a reflection in the x-axis instead of the y-axis. Many students said the transformation was a dilation of factor -1 in the x-axis. While this is correct and was given full marks, this question was an example of a rotation followed by a reflection, which, in general, will be a reflection.

Question 12

a. This was extremely well done, with more than half of students gaining full marks. Not fully answering the question by saying the solution was the point $x = \frac{3}{2}$, $y = -\frac{3}{4}$, $z = \frac{1}{4}$ was penalised ½ of a mark. Almost all students displayed excellent setting out and explained their row operations clearly. Well done!

b. This was the most poorly answered question in the section, and it was clear that many students were short on time when attempting it or leaving it altogether. Numerical errors were more frequent in this part, with only about 20 students solving $q = \frac{1}{2}$ correctly. Many arrived at a value for q, then did not find the line of intersection. This was awarded 2 out of 4 marks.

Section C – Criterion 6

Students generally performed well in this section.

Question 13

a. This question was extremely well done, with nearly all students answering correctly.

b. This question was also well handled. The difficulties students had were managing both the product and chain rule in the term $x\cos(\sin x)$.

Question 14

a. This question challenged most students. The algebra seemed to be the stumbling block. Many students found it challenging to differentiate, which prevented them from demonstrating the validity of the expression.

b. Only a few students managed to obtain full marks for this question. Students who used their original first derivative expression in terms of x to find the second derivative, instead of applying implicit differentiation to the expression from part a), were less likely to arrive at the required result. Students should be reminded to avoid assuming that LHS = RHS before they've proven the result, as this caused confusion and reflects poor mathematical communication.

Question 15

Students who took a systematic approach to finding the relevant information to sketch the graph were most successful in obtaining all available marks. To secure full marks, students needed to include calculations for the first and second derivatives, stationary and non-stationary points of inflection which included their classification, x and y intercepts, and a clear indication of all asymptotes.

Question 16

The implicit differentiation was handled well, and students generally demonstrated a good understanding of finding the normal. A few missed that the normal of $y = 0$ is $x = 0$. There were several students that spent unnecessary time expanding the expression before differentiating. This often led to algebraic errors. A few wasted more time trying to make $\frac{dy}{dx}$ the subject of the expression before substituting (1, 1).

Question 17

a. This question was mostly well answered. The most common errors occurred when students failed to sketch the correct diagram, particularly neglecting the boundaries, most notably the $y = 1$ was neglected.

b. With the correct diagram, students easily found the area. They should be encouraged to consider all available options with these questions as determining the area in relation to the y-axis was easier than the one with respect to the x-axis.

Question 18

Most students performed well on this question. Those who drew clear graphs were more successful in identifying boundaries. There were some students that struggled with correctly applying the volume formula.

Section D – Criterion 7

Question 19

a. Most realised this was a difference of sines. A few mistakes with the integration and errors in simplifying fractions. A few did not actually integrate.

b. Again, most realised this was an integration by parts. Again, a number of silly errors.

   There was some evidence that students just plugged the integral into the calculator and quoted the result.

Question 20.

This proved more problematical than it should have. A considerable minority could not write the integrand in terms of v and instead looked at a peculiar hybrid of x and v. Then to integrate with respect to v these students merely treated x as a constant. A few did not use the given substitution but put $v = e^x$ instead — they were then left with $v\hat{}2/(1+v)$ but completed successfully. Some did not use a substitution at all – they divided out the given expression and then integrated term by term.

Question 21

a. Poorly done. Most could not apply the product rule and collect the terms together as required.

b. Very few made the connection with part a). Some integrated I_n by parts and deduced the required answer effectively, bypassing part a) entirely.

c. Some saw what was needed and worked accordingly. Most did not really see how to proceed.

Question 22

By contrast, this was rather well done. There was a bit of fluffing around with the integration constant and many answers were not simplified very well. But the majority of serious attempts were on the correct track and showed some genuine understanding.

Question 23

This was also reasonably well answered. Most recognised this as a homogeneous equation and made the correct substitution. One common error was to omit the factor 2 on the LHS of the equation; this had the unfortunate consequence that the 'v' integration was not $\frac{1}{v(v-1)}$ but. $\frac{1}{v^2}$; the marks on offer for doing the partial fractions were then lost. Many answers were not taken into simplified form but left in terms of sums/differences of logarithms. Unfortunately, many of the attempts to combine logs made a mess of the process.

Question 24

Lots of evidence that time pressure had cut in by this point. Several valiant attempts to solve part a). Several answers to address part b) used the technique known as "proof by assertion". A number of students solved part c) successfully, which was really pleasing. They demonstrated very good skills with the calculator; they used the machine to solve the simultaneous equations following from the given information and then found what was asked for.

Section E – Criterion 8

This section proved to be a challenge to complete for a lot of students, possibly due to time pressure resulting from the two previous sections.

Question 25

Many candidates were surprisingly unsuccessful with this question. Most knew what they had to do, but the ones who obtained the correct answer were in the minority. Algebraic errors were numerous. There was also evidence of excessive calculator use; marks were deducted in these cases.

Question 26

Probably the least successful question in this section. Most students had no idea how to approach it. 1 mark was awarded for a correct sketch of $arg(z-5)=\frac{2\pi}{3}$. There were only three fully correct responses; two of these used an argand diagram to obtain the answer via geometry, and one used algebra and found the minimum via derivative.

Question 27

a. More successful than Q26, but still mostly unsuccessful. Most candidates could begin the proof, but most did not proceed past the first line. A few successfully simplified the expansion, but then did not realise a double angle formula was required (in some cases) to finish the proof. These cases were awarded 1.5 marks.

b. Largely untouched by most candidates. Very few could successfully find the link to part a). Very few (less than 6) solved this question correctly using part a). Some successfully found the correct equation to solve but were unsure of how to proceed from there. Some students used a guess-and-check method to find a solution. If appropriate algebra was employed beyond this point, marks were generously awarded.

Question 28

a. Nearly all students could handle this question, algebraic errors notwithstanding. Some students substituted both the given solution and its conjugate into the polynomial, but then solved simultaneously on their calculators. Marks were deducted in these cases.

b. Generally quite well done. Occasionally, errors in part a resulted in unfortunate numbers to work with in part b). Some students used calculators excessively – either for expanding factors, or for dividing to obtain the second quadratic factor. Marks were deducted in these cases.

Question 29

Both sketches were quite well handled.

a. The vast majority of candidates sketched a circle; most errors included incorrect location of the centre, or incorrect radius.

b. Again, most students sketched a circle and a shaded angle region. Common errors included an incorrect circle radius, incorrect angle, or poor execution of combining the two. If these parts were successful, the most common following error was incorrect resolving of the vertical line. Some drew a horizontal line, and others drew a vertical line in an incorrect location. Approximately fifteen students had a fully correct sketch, and of these, less than three were awarded the full 6 marks for including correct intersection points.

Question 30

a. Very well done.

b. Fairly well done.

c. If attempted, most students successfully used part b) to start the proof. The rest of the proof was about 50% successful.

d. i. and ii., if attempted, most candidates were quite successful. Some candidates began by assuming the equation to be proved and then rearranged it to obtain the result from c), rather than work the other way around. 0.5 marks were awarded in these cases.

   iii. Mostly unattempted. Most of those who attempted easily achieved 0.5 marks. Less than 10 candidates achieved the full 1 mark here.

Appendix 1: Solutions

Section A

Q1.

If $|2-9x|>6$ we need that either $2-9x>6$ or $2-9x<-6$

(1)

Hence $9x<-4$ or $9x>8$

$$\implies x<-\frac{4}{9} \qquad \text{or} \qquad x>\frac{8}{9}. \qquad (2)$$

Q2.

To prove:

$$\sum_{k=1}^{n} k\times 3^k = \frac{3}{4}\left[(2n-1)\times 3^n+1\right]$$

When $n=1$

$$\text{LHS}=3 \qquad \text{RHS}=\frac{3}{4}[3+1]=3. \qquad (1)$$

Assume true when $n=N$; ie. assume that

$$\sum_{k=1}^{N} k\times 3^k = \frac{3}{4}\left[(2N-1)3^N+1\right]. \qquad (1)$$

Given this, we need to prove statement is true when $n=N+1$, ie.

$$\sum_{k=1}^{N+1} k\times 3^k = \frac{3}{4}\left[(2N+1)3^{N+1}+1\right]. \qquad (1)$$

Now

$$\sum_{k=1}^{N+1} k\times 3^k = \sum_{k=1}^{N} k\times 3^k + (N+1)\times 3^{N+1}$$

$$=\frac{3}{4}\left[(2N-1)3^N+1\right]+(N+1)\times 3^{N+1}=\frac{3}{4}\left[(2N-1)3^N+1+4(N+1)\times 3^N\right]$$

$$=\frac{3}{4}\left[\{(2N-1)+4(N+1)\}\times 3^N+1\right]=\frac{3}{4}\left[(6N+3)\times 3^N+1\right]=\frac{3}{4}\left[(2N+1)\times 3^{N+1}+1\right], \qquad (2)$$

as required.

Hence if the statement is true for $n=N$, it is true for $n=N+1$. As it is true for $n=1$, it holds for all integers N.

(1)

Q3.

The sequence

$$\frac{7n+5}{2n-9} \to \frac{7}{2}$$

as $n \to \infty$ if $\forall \varepsilon > 0$, $\exists N$ such that

$$\left|\frac{7n+5}{2n-9} - \frac{7}{2}\right| < \varepsilon \qquad \text{whenever} \quad n > N. \qquad (1)$$

Now

$$\frac{7n+5}{2n-9} - \frac{7}{2} = \frac{2(7n+5) - 7(2n-9)}{2(2n-9)} = \frac{73}{2(2n-9)}, \qquad (1)$$

Then

$$\frac{73}{2(2n-9)} < \varepsilon \quad \text{if} \quad 2n - 9 > \frac{73}{2\varepsilon} \quad \Longrightarrow \quad n > \frac{73}{4\varepsilon} + \frac{9}{2} \qquad (2)$$

Thus if

$$N = \frac{73}{4\varepsilon} + \frac{9}{2}$$

then

$$\left|\frac{7n+5}{2n-9} - \frac{7}{2}\right| < \varepsilon \quad \text{whenever} \quad n > N \qquad (1)$$

as required.

Q4.

a)

$$\sum_{r=1}^{n}(5r-2)^2 = 25\sum_{r=1}^{n} r^2 - 20\sum_{r=1}^{n} r + 4\sum_{r=1}^{n} 1$$

$$= \frac{25}{6}n(n+1)(2n+1) - 10n(n+1) + 4n \qquad (1)$$

$$= \frac{n}{6}\left[25(n+1)(2n+1) - 60(n+1) + 24\right] = \frac{n}{6}\left[50n^2 + 75n + 25 - 60n - 60 + 24\right]$$

$$= \frac{n}{6}\left\{an^2 + bn + c\right\} \qquad \text{where} \quad a = 50,\ b = 15,\ c = -11. \qquad (2)$$

b) If $\sum_{r=1}^{k}(5r-2)^2 = 94k^2$ then

$$94k^2 = \frac{k}{6}\left\{50k^2 + 15k - 11\right\} \qquad \Longrightarrow \qquad 564k = 50k^2 + 15k - 11 \qquad (1)$$

if $k \neq 0$. Hence

$$50k^2 - 549k - 11 = 0 \quad \Longrightarrow \quad (50k+1)(k-11) = 0 \qquad (1)$$

and so

$$k = 11. \qquad (1)$$

Q5.

a) We note that

$$\frac{1}{r(r+2)} = \frac{1}{2}\left[\frac{1}{r} - \frac{1}{r+2}\right]. \qquad (1)$$

Using this, we can show that

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{1}{2}\left[\frac{1}{1} - \frac{1}{3}\right] + \frac{1}{2}\left[\frac{1}{2} - \frac{1}{4}\right] + \frac{1}{2}\left[\frac{1}{3} - \frac{1}{5}\right] + \dots$$

$$\dots + \frac{1}{2}\left[\frac{1}{n-2} - \frac{1}{n}\right] + \frac{1}{2}\left[\frac{1}{n-1} - \frac{1}{n+1}\right] + \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+2}\right], \qquad (1)$$

$$= \frac{1}{2}\left[\frac{3}{2} - \left(\frac{1}{n+1} + \frac{1}{n+2}\right)\right] = \frac{1}{2}\left[\frac{3}{2} - \frac{2n+3}{(n+1)(n+2)}\right]$$

$$= \frac{1}{2}\left[\frac{3(n^2+3n+2) - 4n - 6}{2(n+1)(n+2)}\right] = \frac{1}{2}\left[\frac{3n^2+5n}{2(n+1)(n+2)}\right] = \frac{n(3n+5)}{4(n+1)(n+2)}. \qquad (2)$$

b) As $n \to \infty$ so the sum tends to $3/4$.

(1)

Then

$$\sum_{r=N+1}^{\infty} \frac{1}{r(r+2)} = \frac{3}{4} - \frac{N(3N+5)}{4(N+1)(N+2)} = \frac{3(N^2+3N+2) - 3N^2 - 5N}{4(N+1)(N+2)} = \frac{2N+3}{2(N+1)(N+2)}.$$

If this is less than $1/10$ it follows that

$$(N+1)(N+2) > 5(2N+3) \quad \Longrightarrow N^2 - 7N - 13 > 0 \quad \Longrightarrow (N - 7/2)^2 > 13 + (49/4) = 101/4$$

Hence

$$N > \frac{7}{2} + \frac{\sqrt{101}}{2} \approx 8.5$$

meaning that $N = 9$ will do.

(2)

Q6.

a) If $f(x) = \ln(1+x)$, then $f' = 1/(1+x)$, $f'' = -1/(1+x)^2$ and $f''' = 2/(1+x)^3$

(1)

so that

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1 \quad \text{and} \quad f'''(0) = 2 \qquad (1)$$

whereupon

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots. \qquad (1)$$

b) Now=

$$\ln\left(\frac{1}{1-2x}\right) = -\ln(1-2x) = -\left[-2x - \frac{1}{2}(2x)^2 - \frac{1}{3}(2x)^3 + \dots\right] = 2x + 2x^2 + \frac{8}{3}x^3 + \dots. \qquad (2)$$

c) If

$$\frac{1}{1-2x} = \frac{3}{2} \quad \Longrightarrow 1 - 2x = \frac{2}{3} \quad \Longrightarrow x = \frac{1}{6} \qquad (1)$$

Then

$$\ln(3/2) \approx 2\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right)^2 + \frac{8}{3}\left(\frac{1}{6}\right)^3 \approx 0.401. \qquad (1)$$

Section B

Q7.

a) If

$$A = \begin{pmatrix} x & 2 \\ -5 & -1 \end{pmatrix} \implies \det A = 10 - x. \qquad (1)$$

If the determinant is -2 then

$$x = 12 \qquad (1)$$

b) With $x = 12$ then

$$A^{-1} = -\frac{1}{2}\begin{pmatrix} -1 & -2 \\ 5 & 12 \end{pmatrix} = \begin{pmatrix} 1/2 & 1 \\ -5/2 & -6 \end{pmatrix}. \qquad (2)$$

Q8.

a) If

$$M = \begin{pmatrix} 6 & -2 \\ -4 & 1 \end{pmatrix}$$

then

$$M^2 = \begin{pmatrix} 6 & -2 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} 6 & -2 \\ -4 & 1 \end{pmatrix} = \begin{pmatrix} 44 & -14 \\ -28 & 9 \end{pmatrix} \qquad (1)$$

while

$$7M + 2I = \begin{pmatrix} 42 & -14 \\ -28 & 7 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 44 & -14 \\ -28 & 9 \end{pmatrix} = M^2 \qquad (1)$$

as required.

b) If P is the point (α, β) then

$$6\alpha - 2\beta = 2k + 8 \qquad \text{and} \qquad -4\alpha + \beta = 2k - 5. \qquad (1)$$

Adding twice the second equation to the first gives

$$-2\alpha = 6k - 2 \implies \alpha = 1 - 3k. \qquad (1)$$

Then $\beta = -1 - 10k$ so that P has co-ordinates

$$(1 - 3k, -1 - 10k) \qquad (1)$$

Q9.

If (x, y) transforms to $(\widehat{x}, \widehat{y})$ then

$$\begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} = B \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \qquad (1)$$

Hence

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} \qquad (1)$$

and so

$$x = \frac{1}{7}(\widehat{x} + 2\widehat{y}), \qquad y = \frac{1}{7}(2\widehat{x} - 3\widehat{y}). \qquad (1)$$

Then the circle is transformed to

$$(\widehat{x} + 2\widehat{y})^2 + (2\widehat{x} - 3\widehat{y})^2 = 9 \times 49 \qquad \Longrightarrow \quad 5\widehat{x}^2 - 8\widehat{x}\widehat{y} + 13\widehat{y}^2 = 441. \qquad (2)$$

Hence the equation of the image is

$$5x^2 - 8xy + 13y^2 = 441.$$

b) As the area of the pre-image is 9π and the deteminant of the matrix B is -7 the area of the image is $9\pi \times 7 = 63\pi$.

(1)

Q10.

a)

$$\Pi_1 : \ x + 2y - 3z = 4, \qquad \Pi_2 : \ 2x + 4y + cz = d.$$

Since $3 + 2(2) - 3(1) = 4$, the point P $(3, 2, 1)$ lies on Π_1.

Since $2(c) + 4(d/4) + c(-2) = d$, the point Q $(c, d/4, -2)$ lies on Π_2.

(2)

b) The planes are parallel if $c = -6$.

(1)

c) With $c = -6$ and $d = 8$ then Q is the point $(-6, 2, -2)$ and as $(-6) + 2(2) - 3(-2) = 4$, it now lies on the plane Π_1 as well.

(2)

d) With the points $(3, 2, 1)$ and $(-6, 2, -2)$ both on plane Π_1, an embedded line is given by

$$(x, y, z) = (3, 2, 1) + \lambda(-9, 0, -3) = (3 - 9\lambda, 2, 1 - 3\lambda). \qquad (1)$$

Hence line is given by

$$x = 3t, \qquad y = 2, \qquad z = t \qquad (1)$$

Q11.

a) The matrix for rotation thrugh an angle θ anticlockwise is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{1}$$

The matrix for reflection in the line $y = x\tan\alpha$ is

$$\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}. \tag{1}$$

Hence with $\theta = \pi/2$ and $\alpha = 3\pi/4$ the combined matrix is

$$T = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2}$$

b) This is a reflection matrix with $\cos 2\alpha = -1$ so that $\alpha = \pi/2$.

(1)

Thus the combination is a reflection in the y-axis.

(1)

Q12.

a) Consider the augmented matrix

$$\begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 1 & 3 & 0 \\ 0 & 4 & 8 & -1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -4 & -1 \end{bmatrix} \tag{2}$$

Back-substituting gives $z = 1/4$, then $y = -3/4$ and $x = 3/2$.

(2)

b) With the amended equations it follows that

$$\begin{bmatrix} 2 & 1 & -1 & 2 \\ 1 & 1 & 1 & q \\ 4 & 5 & 7 & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 1 & 3 & 2q-2 \\ 0 & 3 & 9 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & -1 & 2 \\ 0 & 1 & 3 & 2q-2 \\ 0 & 0 & 0 & 3-6q \end{bmatrix} \tag{2}$$

The equations are only consistent if $q = 1/2$.

Then $y = -1 - 3z$ and $2x = 3 + 4z$ so $x = (3/2) + 2z$.

Hence solution is

$$x = \frac{3}{2} + 2t, \qquad y = -1 - 3t, \qquad z = t. \tag{2}$$

TASMANIAN ASSESSMENT, STANDARDS & CERTIFICATION

Section C

Q13.

a) If $y = \arctan(2x^2)$ then

$$\frac{dy}{dx} = \frac{d/dx(2x^2)}{1+(2x^2)^2} = \frac{4x}{1+4x^4}. \qquad (2)$$

b) If $y = 3^{2x} + x\cos\vartheta = 9^x + x\cos\vartheta$ where $\vartheta = \sin x$ then

$$\frac{dy}{dx} = 9^x \ln 9 + \cos\vartheta - x\sin\vartheta \times \frac{d\vartheta}{dx} \qquad (2)$$

$$= 9^x \ln 9 + \cos\vartheta - x\cos x \sin\vartheta. \qquad (1)$$

Q14.

a) Since $y = \left[x + \sqrt{x^2+1}\right]^p$ it follows that

$$\frac{dy}{dx} = p\left[x+\sqrt{x^2+1}\right]^{p-1} \times \frac{d}{dx}\left(x+\sqrt{x^2+1}\right) = p\left[x+\sqrt{x^2+1}\right]^{p-1}\left(1+\frac{x}{\sqrt{x^2+1}}\right) \qquad (1)$$

$$= p\left[x+\sqrt{x^2+1}\right]^{p-1}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right) = p\frac{\left[x+\sqrt{x^2+1}\right]^p}{\sqrt{x^2+1}} = \frac{py}{\sqrt{x^2+1}}. \qquad (1)$$

Hence

$$\sqrt{1+x^2}\frac{dy}{dx} = py. \qquad (\ddagger)$$

b) Differentiating ‡ gives

$$\sqrt{1+x^2}\frac{d^2y}{dx^2} + \frac{x}{\sqrt{1+x^2}}\frac{dy}{dx} = p\frac{dy}{dx} \qquad (1)$$

Then

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = p\sqrt{1+x^2}\frac{dy}{dx} \qquad (1)$$

Using ‡ again gives

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = p^2y \implies (1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - p^2y = 0. \qquad (1)$$

Q15.

If $y = (x+2)/(x-3)^2$ then

$$\frac{dy}{dx} = \frac{(x-3)^2 - 2(x+2)(x-3)}{(x-3)^4} = \frac{x-3-2x-4}{(x-3)^3} = -\frac{x+7}{(x-3)^3}. \qquad (1)$$

Then

$$\frac{d^2y}{dx^2} = -\left[\frac{(x-3)^3 - 3(x+7)(x-3)^2}{(x-3)^6}\right] = \frac{3(x+7)-(x-3)}{(x-3)^4} = \frac{2(x+12)}{(x-3)^4}. \qquad (1)$$

There is a turning point at $x = -7$, $y = -1/20$ which is a minimum since $y''(-7) > 0$.

(1)

Also $y \to 0$ as $x \to \pm\infty$.

(1)

There is a point of inflection at $x = -12$, $y = \frac{-10}{225} = -\frac{2}{45}$.

(1)

The graph has the following characteristics:

Function negative as $x \to -\infty$. As we increase x so the function is decreasing, has a point of inflection at $(-12, -\frac{2}{45})$ and a minimum at $(-7, -\frac{1}{20})$. The function then increases, cuts the x-axis at $x = -2$, the y-axis at $y = 2/9$ and tends to $+\infty$ as $x \to 3$. For $x > 3$ the function is decreasing with $y \to +\infty$ as $x \to 3$ and $y \to 0$ as $x \to \infty$.

(3)

Q16.

Since

$$4y^2(3x^2 - y^2)^2 = (x^3 + y^3)^4$$

then

$$8yy'(3x^2 - y^2)^2 + 8y^2(3x^2 - y^2)(6x - 2yy') = 4(x^3 + y^3)^3(3x^2 + 3y^2y'). \qquad (2)$$

At $(x, y) = (1, 1)$ we have

$$8y'(4) + 8(2)(6 - 2y') = 4(8)(3 + 3y') \implies 32y' + 96 - 32y' = 96 + 12y'$$

so that

$$y' = 0 \qquad (1)$$

This means that the tangent is parallel to the x-axis and so the normal is given by $y = 1$.

(1)

Q17.

a) The required area is bounded by $y = 0$ for $-1 \le x \le 0$; by $y = x^3$ for $0 \le x \le 1$; by $y = 1$ for $0 \le x \le 1$ and by $y = \sqrt{1+x}$ for $-1 \le x \le 0$.

(2)

b)

$$A = \int_{-1}^{0} \sqrt{1+x}\, dx + \int_{0}^{1} \left(1 - x^3\right)\, dx \qquad (2)$$

$$= \left[\frac{2}{3}(1+x)^{3/2}\right]_{-1}^{0} + \left[x - \frac{1}{4}x^4\right]_{0}^{1}$$

$$= \frac{2}{3} + \frac{3}{4} = \frac{17}{12}. \qquad (2)$$

Q18.

The given curves intersect at $(0,0)$ and $(8,4)$; for $0 < x < 8$ then $\frac{x^2}{16} < x^{2/3}$.

(2)

Now rotating area about the x-axis gives

$$V_1 = \pi \int_0^8 \left\{x^{4/3} - \frac{1}{256}x^4\right\}\, dx \qquad (1)$$

$$= \pi \left[\frac{3}{7}x^{7/3} - \frac{x^5}{5 \times 256}\right]_0^8 = \pi \left[\frac{3}{7} \times 128 - \frac{128}{5}\right]$$

$$= 128\pi \left(\frac{3}{7} - \frac{1}{5}\right) = \frac{128 \times 8}{35}\pi. \qquad (2)$$

Rotating about the y-axis gives

$$V_2 = \pi \int_0^4 \left[16y - y^3\right]\, dy = \pi \left[8y^2 - \frac{1}{4}y^4\right]_0^4 = \pi(128 - 64) = 64\pi. \qquad (2)$$

Hence

$$\frac{V_1}{V_2} = \frac{128 \times 8}{35 \times 64} = \frac{16}{35} \implies V_1 = \frac{16}{35}V_2. \qquad (1)$$

Section D

Q19.

a)

$$\int_0^{\pi/2} \sin 2x \cos 5x \, dx = \frac{1}{2}\int_0^{\pi/2} (\sin 7x - \sin 3x) \, dx \qquad (1)$$

$$= \frac{1}{2}\left[-\frac{1}{7}\cos 7x + \frac{1}{3}\cos 3x\right]_0^{\pi/2} = -\frac{1}{2}\left(\frac{1}{3} - \frac{1}{7}\right) = -\frac{2}{21}. \qquad (1)$$

b)

$$\int_1^e x^2 \ln x \, dx = \left[\frac{1}{3}x^3 \ln x\right]_1^e - \frac{1}{3}\int_1^e x^2 \, dx \qquad (1)$$

$$= \frac{1}{3}e^3 - \left[\frac{1}{9}x^3\right]_1^e = \frac{1}{3}e^3 - \frac{1}{9}e^3 + \frac{1}{9} = \frac{1}{9}\left(2e^3 + 1\right). \qquad (2)$$

Q20.

If

$$I = \int_0^1 \frac{e^{3x}}{1 + e^x} dx,$$

let $v = 1 + e^x$ so $\frac{dv}{dx} = e^x$. When $x = 0$, $v = 2$ and when $x = 1$, $v = 1 + e$.

(1)

Then

$$I = \int_2^{1+e} \frac{(v-1)^2}{v} \, dv \qquad (2)$$

$$= \int_2^{1+e} \left(v - 2 + \frac{1}{v}\right) dv = \left[\frac{1}{2}v^2 - 2v + \ln v\right]_2^{1+e}, \qquad (1)$$

$$= \frac{1}{2}(1+e)^2 - 2(1+e) + \ln(1+e) - 2 + 4 - \ln 2 = \frac{1}{2}e^2 - e + \frac{1}{2} + \ln\left(\frac{1+e}{2}\right) = \frac{1}{2}(e-1)^2 + \ln\left(\frac{1+e}{2}\right). \qquad (1)$$

Q21.

a)

$$\frac{d}{dx}\left\{2x^n(1-x)^{3/2}\right\} = 2nx^{n-1}(1-x)^{3/2} - 3x^n\sqrt{1-x}$$

$$= \{2n(1-x) - 3x\}\, x^{n-1}\sqrt{1-x} = \{2n - (2n+3)x\}\, x^{n-1}\sqrt{1-x} \qquad (1)$$

b) If $I_n \equiv \int_0^1 x^n\sqrt{1-x}\,dx$ then integrating the above result over the interval $[0,1]$ gives

$$\left[2x^n(1-x)^{3/2}\right]_0^1 = 2nI_{n-1} - (2n+3)I_n \qquad (2)$$

Since

$$\left[2x^n(1-x)^{3/2}\right]_0^1 = 0$$

it follows that

$$0 = 2nI_{n-1} - (2n+3)I_n \implies I_n = \frac{2n}{2n+3}I_{n-1} \qquad (\natural)$$

(1)

c) Using ($\natural$) repeatedly with $n = 3, 2, 1$ gives

$$I_3 = \frac{6}{9}I_2 = \frac{2}{3}\times\frac{4}{7}I_1 = \frac{8}{21}\times\frac{2}{5}I_0 = \frac{16}{105}I_0. \qquad (2)$$

Since

$$I_0 = \int_0^1 \sqrt{1-x}\,dx = \left[-\frac{2}{3}(1-x)^{3/2}\right]_0^1 = \frac{2}{3} \qquad (1)$$

we conclude that $I_3 = 32/315$.

Q22.

If

$$x^2\frac{dy}{dx} - y = 1 \qquad \text{then} \quad x^2\frac{dy}{dx} = 1 + y$$

so

$$\int \frac{dy}{1+y} = \int \frac{dx}{x^2} \qquad (2)$$

Then

$$\ln(1+y) = \ln C - \frac{1}{x} \implies 1 + y = Ce^{-1/x} \qquad (1)$$

If $y(1) = 1$ then $2 = C/e$ so $C = 2e$

(1)

Then

$$y = 2e^{1-\frac{1}{x}} - 1 \qquad (1)$$

Q23.

For the differential equation

$$2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2},$$

put $y = vx$ so that

$$(1)$$

$$2\left[v + x\frac{dv}{dx}\right] = v + v^2 \implies 2x\frac{dv}{dx} = v^2 - v, \qquad (1)$$

Hence

$$2\int \frac{dv}{v(v-1)} = 2\int \left(\frac{1}{v-1} - \frac{1}{v}\right) dv = \int \frac{dx}{x} \qquad (1)$$

$$2\left[\ln(v-1) - \ln v\right] = \ln C + \ln x \implies \ln\left(\frac{v-1}{v}\right)^2 = \ln Cx \implies \left(\frac{v-1}{v}\right)^2 = Cx \qquad (2)$$

Hence

$$\frac{v-1}{v} = \pm C\sqrt{x}$$

and as $v(1) = 2$ so

$$\frac{v-1}{v} = \frac{1}{2}\sqrt{x} \implies \frac{1}{v} = 1 - \frac{1}{2}\sqrt{x} = \frac{2-\sqrt{x}}{2} \qquad (1)$$

$$\therefore\ v = \frac{2}{2-\sqrt{x}} \implies y = vx = \frac{2x}{2-\sqrt{x}}. \qquad (1)$$

Q24.

The differential equation is

$$\frac{dW}{dt} = \frac{r}{100} W(100 - W).$$

a) **EITHER**

$$\int \frac{dW}{W(100-W)} = \frac{r}{100}\int dt \implies \int \left[\frac{1}{100-W} + \frac{1}{W}\right] dW = r\int dt \qquad (1)$$

$$\ln(100-W) - \ln W = \ln K - rt \implies \ln\left(\frac{100-W}{W}\right) = \ln(Ke^{-rt}) \implies \frac{100-W}{W} = Ke^{-rt} \qquad (1)$$

If $W = W_0$ when $t = 0$ then

$$\frac{100-W}{W} = \frac{100-W_0}{W_0}e^{-rt} \implies \frac{100}{W} = 1 + \frac{100-W_0}{W_0}e^{-rt} \qquad (1)$$

so

$$W = \frac{100W_0}{W_0 + (100-W_0)e^{-rt}} \qquad (1)$$

OR

If we define $\Xi = W_0 + (100 - W_0)e^{-rt}$ then $d\Xi/dt = -r(100 - W_0)e^{-rt} = r(W_0 - \Xi)$

(1)

With

$$W(t) = \frac{100W_0}{\Xi}$$

$$\frac{dW}{dt} = -\frac{100W_0}{\Xi^2}\frac{d\Xi}{dt} = \frac{100rW_0}{\Xi^2}(\Xi - W_0) = rW\left(1 - \frac{W_0}{\Xi}\right) = rW\left(1 - \frac{W}{100}\right) = \frac{r}{100}W(100-W) \qquad (2)$$

Also this formula gives $W(0) = W_0$ as required.

(1)

b) As $t \to \infty$ so $e^{-rt} \to 0$ so

$$W \to \frac{100W_0}{W_0} = 100$$

Hence as $t \to \infty$ all the burrows become occupied.

(1)

c) The easiest way to proceed is to measure time from when the first visit occurs. Then we have $W = W_0 = 40$ when $t = 0$.

Given that $W = 60$ when $t = 3$ so

$$60 = \frac{4000}{40 + 60e^{-3r}} \implies 40 + 60e^{-3r} = \frac{200}{3} \implies e^{-3r} = \frac{80}{180} = \frac{4}{9} \implies r = \frac{1}{3}\ln(9/4) \qquad (1)$$

When $W = 90$ then

$$90 = \frac{4000}{40 + 60e^{-rt}} \implies 40 + 60e^{-rt} = \frac{400}{9} \implies e^{-rt} = \frac{40}{540} = \frac{2}{27} \qquad (1)$$

whence

$$rt = \ln(27/2) \implies t = \frac{3\ln(27/2)}{\ln(9/4)} \approx 9.63 \qquad (1)$$

Thus 90% occupancy achieved after $9.63 + 3 \approx 12.63$ months after the start of the experiment.

Section E

Q25.

$$\frac{1}{z}+\frac{1}{w}=\frac{1}{3-4i}+\frac{1}{2-i}=\frac{3+4i}{25}+\frac{2+i}{5}=\frac{13+9i}{25} \qquad (2)$$

Then

$$v=\frac{25}{13+9i}=\frac{25(13-9i)}{169+81}=\frac{25(13-9i)}{250}=\frac{1}{10}(13-9i). \qquad (1)$$

Q26.

If $\arg(z-5)=2\pi/3$ then $z-5=re^{2\pi i/3}$ and

$$z=5+\frac{1}{2}r(-1+i\sqrt{3})=(5-\frac{1}{2}r)+ir\frac{\sqrt{3}}{2} \qquad (1)$$

Then

$$|z|^2=(5-\frac{1}{2}r)^2+\frac{3}{4}r^2=r^2-5r+25 \qquad (1)$$

$$=\left(r-\frac{5}{2}\right)^2+\frac{75}{4} \qquad (1)$$

Hence

$$|z|_{\min}=\sqrt{\frac{75}{4}}=\frac{5}{2}\sqrt{3}. \qquad (1)$$

Q27.

a) If $z=e^{i\theta}$ then $z+z^{-1}=2\cos\theta$ and $z^2+z^{-2}=e^{2i\theta}+e^{-2i\theta}=2\cos 2\theta$

(1)

Hence

$$z^2-z-\frac{1}{z}+\frac{1}{z^2}=2\cos 2\theta-2\cos\theta=4\cos^2\theta-2\cos\theta-2. \qquad (1)$$

b) If $z^4-z^3+2z^2-z+1=0$ then $z^2-z-\frac{1}{z}+\frac{1}{z^2}=-2$

(1)

Using part (a) we conclude that

$$4\cos^2\theta-2\cos\theta=0 \implies \cos\theta(2\cos\theta-1)=0 \implies \cos\theta=0 \text{ or } \frac{1}{2} \qquad (1)$$

Thus

$$\theta=\pm\frac{\pi}{2} \text{ or } =\pm\frac{\pi}{6} \qquad (1)$$

Therefore

$$z=\pm i \quad \text{or} \quad z=\frac{1}{2}(1\pm i\sqrt{3}) \qquad (1)$$

Q28.

a) If $z=-2-3i$ then $z^2=-5+12i$, $z^3=46-9i$ and $z^4=-119-120i$.

(1)

The equation becomes

$$-119-120i+46-9i+8(-5+12i)+\alpha(-2-3i)+\beta=0$$

Real parts give

$$2\alpha-\beta=-113 \quad (1)$$

Imaginary parts give

$$3\alpha=-33 \implies \alpha=-11 \quad (1)$$

so

$$\beta=91 \quad (1)$$

b) If $z=-2-3i$ is a root of $P(z)=0$ then so is $z=-2+3i$

(1)

Then $(z-(-2+3i))(z-(-2-3i))=z^2+4z+13$ is a factor of $P(z)$

(1)

By long division

$$z^4+z^3+8z^2-11z+91=(z^2+4z+13)(z^2-3z+7) \quad (1)$$

and if $z^2-3z+7=0$ then $z=(3\pm i\sqrt{19})/2$.

Hence the roots of the quartic $P(z)=0$ are

$$z=-2\pm 3i, \qquad \frac{1}{2}\left(3\pm i\sqrt{19}\right). \quad (1)$$

Q29.

a) If $z=x+iy$ then

$$|\mathrm{Im}(z-3i)|^2+|\mathrm{Re}(z-4i)|^2=(y-3)^2+x^2=9 \quad (1)$$

This is a circle centre $z=3i$ of radius 3.

(1)

b)

$|z-4-2i|\le 2:$ inside of circle of radius 2 and centre $4+2i$ (1)

$0\le \arg(z-2-2i)\le \pi/4:$ the line joining $2+2\text{i}$ to z is inclined to real axis at angle between 0 and $\pi/4$ (1)

$|z-4|\le|z-6|:$ point is closer to 4 than 6 so that $\mathrm{Re}(z)\le 5$ (1)

Region is the interior of the circle centre $4+2i$ of radius 2 which is bounded to the right by the line Re(z)=5, below by Im(z)=2, to the left by the line through $2+2i$ inclined at $\pi/4$ to the real axis and which meets the circle again at $4+4i$ and above by the small arc of the circumference.

(3)

Q30.

a) If $z^3 = 1$ then $z = 1$ or $z = e^{i\theta}$ where $\theta = \pm 2\pi/3$.

(2)

b) If $\omega = e^{2\pi i/3}$ then $\omega^2 = e^{4\pi i/3} = e^{-2\pi i/3}$
If $\omega = e^{-2\pi i/3}$ then $\omega^2 = e^{-4\pi i/3} = e^{2\pi i/3}$
Hence if one non-real root is ω, the other is ω^2.

(1)

c)

$$1 + \omega + \omega^2 = 1 + e^{2\pi i/3} + e^{-2\pi i/3} = 1 + 2\cos(2\pi/3) = 0 \qquad (1)$$

d)
i)

$$\frac{\omega}{1+\omega} = \frac{\omega}{-\omega^2} = -\frac{1}{\omega} \qquad (1)$$

ii)

$$\frac{\omega^2}{1+\omega^2} = \frac{\omega^2}{-\omega} = -\omega \qquad (1)$$

iii)

$$\left(\frac{\omega}{1+\omega}\right)^k + \left(\frac{\omega^2}{1+\omega^2}\right)^k = \left(-\omega^{-1}\right)^k + (-\omega)^k = (-1)^k \left\{\omega^k + \omega^{-k}\right\}$$
$$= (-1)^k \left\{e^{2k\pi i/3} + e^{-2k\pi i/3}\right\} = 2(-1)^k \cos(2k\pi/3) \qquad (1)$$