OFFICE OF TASMANIAN ASSESSMENT, STANDARDS & CERTIFICATION

External Assessment 2021

Attach your candidate label here

MATHEMATICS SPECIALISED

MTS415118

Pages	16
Questions	30
Information Sheet	1
Answer Booklets	5

Reading time: 15 minutes – you may begin writing during this time

Suggested working time: 3 hours

Instructions

- There are **five (5)** sections to this exam paper.
- Attempt **all questions** and **all parts** within each question.
  - You must show the methods you used to solve questions to receive full marks.
- Answer each section in a **separate answer booklet**.
- Approved calculators and all their functions may be used.
- All answers must be written in **English**.
- You **must** make sure your answers address:
  - Criterion 4 solve problems and use techniques involving finite and infinite sequences and series.
  - Criterion 5 solve problems and use techniques involving matrices and linear algebra.
  - Criterion 6 use differential calculus and apply integral calculus to areas and volumes.
  - Criterion 7 use techniques of integration and solve differential equations.
  - Criterion 8 solve problems and use techniques involving complex numbers.

Guide to Exam Structure

	Questions available	How many questions to answer	Suggested working time	Marks available
Section A	6	6	36 minutes	36
Section B	6	6	36 minutes	36
Section C	6	6	36 minutes	36
Section D	6	6	36 minutes	36
Section E	6	6	36 minutes	36
Total	**30**	**30**	**180 minutes (3 hours)**	**180**

Additional Instructions for Candidates

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you only show your answers you will get few if any marks.

You are permitted to bring into the examination room: pens, pencils, highlighters, erasers, sharpeners, rulers, a protractor, set-squares, aids for curve sketching and any approved scientific or graphics or CAS calculator (memory may be retained). Unless instructed otherwise, calculators may be used to their full capacity when undertaking this examination.

Section A

Attempt **all** questions in this section.

You **must** show the method and workings you use to solve a question.

Use a **separate answer booklet** for this section.

This section assesses **Criterion 4**.

Question 1

a) If 2 marks

$$v_k = \frac{1}{5}(k-1)k(k+1)(k+2)(k+3),$$

verify that

$$k(k+1)(k+2)(k+3) = v_{k+1} - v_k.$$

b) Evaluate 4 marks

$$\sum_{k=1}^{n} k(k+1)(k+2)(k+3).$$

Question 2

Determine whether the following sequences $\{u_n\}$ converge or diverge.

Give a brief justification for your assertions and state the limits of any sequences that do converge.

a) $u_n = \cos(n)$ 2 marks

b) $u_n = \ln(2n+3) - \ln(3n+2)$ 3 marks

c) $u_n = \tan^{-1}[(-1)^n \sqrt{n}]$ 2 marks

Question 3

For what values of x does the infinite geometric progression 4 marks

$$3 + (x-2) + \frac{1}{3}(x-2)^2 + \frac{1}{9}(x-2)^3 + \ldots$$

converge?

Section A continues

Question 4

a) Use formal methods to prove that the sequence $\left\{\dfrac{n^2-10}{2n+7}\right\}$ diverges to infinity. 4 marks

b) Determine the smallest positive integer N for which 3 marks

$$\frac{N^2-10}{2N+7} > 100.$$

Question 5

Use the method of induction to prove that 7 marks

$$1+2x+3x^2+...+nx^{n-1} = \frac{1-(n+1)x^n+nx^{n+1}}{(1-x)^2} \qquad (x \neq 1).$$

Question 6

The Maclaurin expansion of the function $e^{mx}-(1+4x)^n$ is $-4x^2+...$ 5 marks

Determine the values of the constants m and n.

Section B

Attempt **all** questions in this section.

You **must** show the method and workings you use to solve a question.

Use a **separate answer booklet** for this section.

This section assesses **Criterion 5**.

Question 7

a) Sketch the image of the unit square under a shear of size 3 parallel to the y-axis followed by a dilation of factor 4 parallel to the x-axis. 4 marks

b) What is the area of the image? 2 marks

Question 8

For the matrix $\boldsymbol{A} = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$

show that:

a) $\boldsymbol{A}^2 = 3\boldsymbol{A} - 2\boldsymbol{I}$ 2 marks

and

b) $2\boldsymbol{A}^{-1} = 3\boldsymbol{I} - \boldsymbol{A}.$ 2 marks

Question 9

Suppose that Q is the plane defined by the equation $4x - 3y + 6z = 36.$

a) Show that the point $A\,(3, 2, 5)$ lies on $Q.$ 1 mark

b) Determine the equation of the line that joins A to the point $B\,(1, 2, -3).$ 3 marks

c) Find the equation of the plane parallel to Q that contains the point $B.$ 2 marks

Question 10

A curve C is rotated by $\pi/4$ **clockwise** about the origin and then reflected in the line $y = \dfrac{x}{\sqrt{3}}.$

a) Show that if the point (x, y) is transformed to (X, Y) then 4 marks

$(1+\sqrt{3})y - (\sqrt{3}-1)x = 2\sqrt{2}X$ and $(1+\sqrt{3})x + (\sqrt{3}-1)y = 2\sqrt{2}Y$

b) If the equation of the image is $xy = 1,$ determine the equation of C. 3 marks

Section B continues

Question 11

A system of linear equations for three unknowns is given by

$$x - 2y + z = 7$$

$$2x + y - 2z = 1$$

$$-x + \alpha y + 2z = \beta$$

a) What are the restrictions on α and β if this system has no solution? 5 marks

b) If the system has the unique solution $(x, y, z) = (3, -1, 2)$ determine the relationship between α and β. 2 marks

Question 12

Let the matrix $\boldsymbol{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in which a, b, c and d are real numbers.

The *trace* of $\boldsymbol{M}$ written $Tr(\boldsymbol{M})$ is defined to be the sum of the leading diagonal entries so $Tr(\boldsymbol{M}) = a + d$.

Det $\boldsymbol{M}$ is defined to be the determinant of $\boldsymbol{M}$.

a) Prove that $Tr(\boldsymbol{M}^2) = [Tr(\boldsymbol{M})]^2 - 2\det(\boldsymbol{M})$. 3 marks

b) If $\boldsymbol{M}^2 = \boldsymbol{I}$, but $\boldsymbol{M} \neq \pm\boldsymbol{I}$ find the trace and determinant of $\boldsymbol{M}$. 3 marks

Section C

Attempt **all** questions in this section.

You **must** show the method and workings you use to solve a question.

Use a **separate answer booklet** for this section.

This section assesses **Criterion 6**.

Question 13

a) Differentiate $x^4 e^{-2x}$ with respect to x. 2 marks

b) Determine 2 marks

$$\frac{d}{dx}\left\{\sin^{-1}\left(\sqrt{x}\right)\right\}.$$

Question 14

Determine the two points of inflection of the function $y(x) = e^{x(2-x)}$. 4 marks

Question 15

Consider the function

$$f(x) = \frac{x^2 - 4x + 20}{x - 2}.$$

a) Determine the intercepts and critical points of $f(x)$. 4 marks

b) Find the behaviour of $f(x)$ as $x \to \pm\infty$. 1 mark

c) Sketch the graph of $y = f(x)$. 4 marks

Question 16

Consider the curve given implicitly by $(x^2 + y^2)^3 = 8x^2 y^2$. 5 marks

Determine the equations of the tangent and normal at the point $(1,1)$.

Section C continues

Question 17

a) Determine the area of the region R enclosed by the parabola $y = x^2$, the hyperbola $xy = 8$ and the line $y = 1$. 4 marks

b) Determine the volume generated if R is rotated about the x-axis. 3 marks

Question 18

Given that μ is a positive constant, the area A is the region in the first quadrant bounded by the parabola $y = \mu(4 - x^2)$ and the co-ordinate axes. 7 marks

The area A is rotated about the x-axis to form a solid of volume V_1.

The same area is next rotated about the y-axis to generate another solid of volume V_2.

If $V_1 = V_2$ find the value of μ.

Section D

Attempt **all** questions in this section.

You **must** show the method and workings you use to solve a question.

Use a **separate answer booklet** for this section.

This section assesses **Criterion 7**.

Question 19

a) Determine 3 marks

$$\int \cos 3\theta \cos 2\theta d\theta$$

b) Evaluate 3 marks

$$\int_0^1 xe^{-3x} dx.$$

Question 20

a) Solve 3 marks

$$\frac{dy}{dx} = e^{-y}(2x-4), \qquad y(5)=0$$

b) For what values of x is your solution valid? 2 marks

Question 21

On a certain nature reserve there are initially 10 pairs of nesting black cockatoos. One theory suggests that the number $N,$ of the nesting pairs after t years will satisfy the differential equation

$$\frac{dN}{dt} = \frac{N}{180}(100-N).$$

a) Show that initially the rate of increase of N is 5 per year. 1 mark

b) Interpret what happens as $N \to 100.$ 1 mark

c) Determine $N(t).$ 4 marks

d) After how many years does the population of black cockatoos reach 90% of its maximum? 2 marks

Section D continues

Question 22

Solve the differential equation 5 marks

$$x\frac{dy}{dx} = y + \frac{y^2}{x}, \qquad y(1) = 1.$$

Question 23

a) Use the substitution $v = a - x$ to prove that 2 marks

$$\int_0^a f(x)dx = \int_0^a f(a-x)dx.$$

Define the integral $$I = \int_0^{\pi} \frac{x\sin x}{3+\cos^2 x}dx.$$

b) i. Use the result of (a) to demonstrate that 2 marks

$$2I = \pi\int_0^{\pi} \frac{\sin x}{3+\cos^2 x}dx.$$

ii. Hence, or otherwise, evaluate I . 3 marks

Question 24

Use the substitution $u = 1 - \frac{1}{x}$ to evaluate 5 marks

$$\int_{9/8}^{4/3} \frac{dx}{x^{3/2}\sqrt{x-1}}.$$

Section E

Attempt **all** questions in this section.

You **must** show the method and workings you use to solve a question.

Use a **separate answer booklet** for this section.

This section assesses **Criterion 8**.

Question 25

Show that if $z = \sqrt{3} - i$ then z^3 is purely imaginary. 3 marks

Question 26

a) Express the complex number 4 marks

$$w = \frac{1+i}{\sqrt{3}+i}$$

in both Cartesian and polar forms.

b) Hence, or otherwise, deduce an exact value for $\tan\left(\frac{\pi}{12}\right)$. 3 marks

Question 27

If z_1 and z_2 are any two complex numbers prove that 3 marks

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left\{|z_1|^2 + |z_2|^2\right\}.$$

Question 28

a) Determine the real numbers α and β if $z - 1 + 2i$ is one factor of 4 marks

$$p(z) = z^4 + \alpha z^3 + 18z^2 + \beta z + 35$$

b) Hence, or otherwise, determine all the solutions of $p(z) = 0$. 4 marks

Question 29

a) Sketch the region R of the complex plane defined by 4 marks

$$|z-1| \geq |z-i| \quad \text{and} \quad |z - 2(1+i)| \leq 1$$

b) Determine the modulus of the complex number $z \in R$ for which $\arg(z)$ is greatest. 3 marks

Section E continues

Question 30

a) Verify that 1 mark

$$(z - e^{i\varphi})(z - e^{-i\varphi}) \equiv z^2 - 2z\cos\varphi + 1.$$

b) Solve $z^5 + 1 = 0$. Give your answers in polar form $re^{i\varphi}$ with $-\pi < \varphi \leq \pi$. 4 marks

c) **Given** that 3 marks

$$1 - z + z^2 - z^3 + z^4 = \frac{z^5 + 1}{z + 1}$$

express

$$z^4 - z^3 + z^2 - z + 1$$

as the product of two quadratic factors with real coefficients.

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End of Exam

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