OFFICE OF TASMANIAN ASSESSMENT, STANDARDS & CERTIFICATION

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External Assessment 2022

MATHEMATICS SPECIALISED

MTS415118

Pages	16
Questions	30
Answer Booklets	5
Information Sheet	1

Preparation time for this exam: 15 minutes

Suggested working time: 3 hours

Instructions:

- There are **five (5)** sections to this exam paper.
- Answer **all** questions and **all** items within each question.
- You **must** show the methods you used to solve questions to receive full marks.
- Answer each section in a **separate answer booklet**.
- Approved calculators and all their functions may be used.
- The exam is **three (3)** hours in length. The suggested working time is provided in the instructions for each section.
- The Mathematics Specialised Information Sheet can be used throughout this exam.
- All answers must be written in **English**.
- You **must** make sure your answers address:
  - Criterion 4 solve problems and use techniques involving finite and infinite sequences and series
  - Criterion 5 solve problems and use techniques involving matrices and linear algebra
  - Criterion 6 use differential calculus and apply integral calculus to areas and volumes
  - Criterion 7 use techniques of integration and solve differential equations
  - Criterion 8 solve problems and use techniques involving complex numbers.

Guide to Exam Structure

	Questions available	Questions to answer	Suggested working time	Marks available
Section A	6	6	36 minutes	36
Section B	6	6	36 minutes	36
Section C	6	6	36 minutes	36
Section D	6	6	36 minutes	36
Section E	6	6	36 minutes	36
Totals	**30**	**30**	**180 minutes (3 hours)**	**180**

Additional Exam Instructions

Markers will look at your presentation of answers and at the arguments leading to answers when determining your result on each criterion.

You must show the method used to solve a question. If you only show your answers you will get few if any marks.

You are permitted to bring into the exam room:

- pens
- pencils
- highlighters
- erasers
- sharpeners
- rulers
- a protractor
- set-squares
- aids for curve sketching
- any approved scientific or graphics or CAS calculator (memory may be retained).

Unless instructed otherwise, calculators may be used to their full capacity when undertaking this exam.

Section A

- Attempt **all** questions in this section.
- It is suggested that you spend **approximately 36 minutes** on this section.
- You **must** show the method and workings you use to solve a question.
- Use a **separate answer booklet** for this section.
- This section assesses **Criterion 4.**

Question 1

Solve the inequality $|3-2x|>5$. (3 marks)

Question 2

The sum of the first and third terms of a geometric sequence is 15. The second term, which is smaller than the first term, is 6. Determine the sum to infinity of the series. (5 marks)

Question 3

Use mathematical induction to prove that

$$\frac{1}{1\times 2\times 3}+\frac{1}{2\times 3\times 4}+....+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)} \quad \text{for } n\geq 1.$$ (7 marks)

Question 4

a) Use the method of differences to determine

$$\sum_{k=1}^{n}\frac{1}{k(k+2)}.$$ (5 marks)

b) What is the smallest value of N for which

$$\sum_{k=N}^{\infty}\frac{1}{k(k+2)}<0.1\text{ ?}$$ (3 marks)

Section A continues

Question 5

a) Use formal methods to prove that the sequence $\left\{\frac{(-1)^n + n^2}{3+2n^2}\right\}$ converges to $\frac{1}{2}$ as $n \to \infty$.

(5 marks)

b) Discuss whether the sequence $\left\{\frac{1+(-1)^n n^2}{3+2n^2}\right\}$ converges as $n \to \infty$.

(2 marks)

Question 6

a) Derive the first three terms in the Maclaurin series of $\ln(1+x)$. (4 marks)

b) Hence deduce the first three terms in the Maclaurin series for

$$\ln\left(1-2x+x^2\right).$$

(2 marks)

Section B

- Attempt **all** questions in this section.
- It is suggested that you spend **approximately 36 minutes** on this section.
- You **must** show the method and workings you use to solve a question.
- Use a **separate answer booklet** for this section.
- This section assesses **Criterion 5.**

Question 7

If the matrix

$$\mathbf{A} = \begin{pmatrix} -3 & 11 \\ -2 & 5 \end{pmatrix}$$

and if $\mathbf{I}$ is the 2×2 identity matrix, show that $\mathbf{A}^2 - 2\mathbf{A} + 7\mathbf{I} = 0.$ (3 marks)

Question 8

Suppose that $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are 2×2 matrices with

$$\mathbf{A} = \begin{pmatrix} -1 & -1 \\ -2 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

Determine the matrix $\mathbf{C}$ if $\mathbf{CA} = \mathbf{B}$. (3 marks)

Question 9

The circle $x^2 + y^2 = 1$ is transformed by a dilation by a factor of 3 in the $y-$ direction followed by a shear by a factor of 2 in the $x-$ direction.

a) Determine the equation of the image. (5 marks)

b) What is the area of the image? (1 mark)

Section B continues

Question 10

Consider the system of equations

$$\begin{aligned} 2x + 3y + 2z &= 10 \\ 3x + 4y - z &= 4 \\ x + y + az &= b \end{aligned}$$

a) For what values of the constants a and b are there

i. infinitely many solutions?

ii. no solutions? (5 marks)

b) Solve the equations in the case when $a = -2$ and $b = -5$. (2 marks)

Question 11

A transformation M is obtained by first rotating anti-clockwise about the origin through an angle $\frac{\pi}{6}$ and then reflecting in the line $y = \frac{x}{\sqrt{3}}$.

a) Determine the matrix for M. (3 marks)

b) Sketch the image of the unit square when it is transformed by M. (3 marks)

c) What single reflection would have the same effect as the transformation M? (2 marks)

Question 12

The points $P\ (2,7,-3)$, $Q\ (1,\alpha,0)$ and $R\ (\beta,4,6)$ form a straight line.

a) Determine the values of α and β. (3 marks)

b) A plane Π with equation $x + y - 3z = C$ contains the point P.

Write down the value of the constant C. (1 mark)

c) Determine the equation of the plane parallel to Π that passes through Q. (2 marks)

d) A line L through a point $S\ (1,-1,\gamma)$ has the equation

$$(x, y, z) = (1, -1, \gamma) + t(1, 2, \eta)$$

where γ and η are constants.

If this line is embedded in Π determine the values of γ and η. (3 marks)

Section C

- Attempt **all** questions in this section.
- It is suggested that you spend **approximately 36 minutes** on this section.
- You **must** show the method and workings you use to solve a question.
- Use a **separate answer booklet** for this section.
- This section assesses **Criterion 6.**

Question 13

Differentiate the following with respect to x:

a) $\cos\left(x^2\right)+\cos^2 x$. (2 marks)

b) 2^{2x}. (2 marks)

c) $\arcsin(e^{-x})$. (2 marks)

Question 14

If $4y^2+1=3x^2$ prove that

$$y^3\frac{d^2y}{dx^2}=-\frac{3}{16}\ .$$ (5 marks)

Question 15

The point P with co-ordinates $x=ap^2$, $y=2ap$ lies on the parabola $y^2=4ax$.

a) Show that the equation of the tangent to the parabola at P is

$$py=x+ap^2.$$ (3 marks)

b) If Q is another point on the same parabola with co-ordinates $x=aq^2$, $y=2aq$, what is the relationship between the parameters p and q if the tangents at the points P and Q are perpendicular? (1 mark)

c) If the parameter p is varied and q changes so that the two tangents at P and Q remain perpendicular, show that the tangents meet on a line that is parallel to the $y-$axis. (3 marks)

Section C continues

Section C continued

Question 16

Consider the function

$$f(x)=\frac{x^2-5x+7}{x-2}\ .$$

a) Determine the locations of any zeros and critical points of $f(x)$. (4 marks)

b) Sketch the graph of $y=f(x)$. (3 marks)

Question 17

a) Sketch the two curves $x=10-y^2$ and $x=(y-2)^2$ together with the region R contained between them. (2 marks)

b) Determine the area of R. (4 marks)

Question 18

The area bounded by the two quadratic functions $y=x(4-x)$ and $y=2x(4-x)$ is rotated completely about the $x-$axis.

Find the volume of the solid of revolution formed. (5 marks)

Section D

- Attempt **all** questions in this section.
- It is suggested that you spend **approximately 36 minutes** on this section.
- You **must** show the method and workings you use to solve a question.
- Use a **separate answer booklet** for this section.
- This section assesses **Criterion 7**.

Question 19

Evaluate

a) $\int_0^1 \frac{dx}{1+3x^2}$. (3 marks)

b) $\int_0^1 \frac{dx}{4-x^2}$. (4 marks)

Question 20

Use double angle formulae to determine

$$\int_0^{\pi/8} \frac{d\theta}{1+\cos 4\theta}.$$ (4 marks)

Question 21

Determine

$$\int e^{-x} \sin 2x \, dx.$$ (6 marks)

Question 22

Solve the differential equation

$$\frac{dy}{dx} = \frac{x(e^{x^2}+2)}{6y^2}$$ given that $y=1$ when $x=0$. (5 marks)

Section D continues

Question 23

Determine the solution of the equation

$$\frac{dy}{dx} = \frac{2x^3 + y^3}{3xy^2} \qquad \text{given that } y = 2 \text{ when } x = 1.$$

(7 marks)

Question 24

A hot kettle is placed in a room which is at temperature Θ degrees Celsius.

After a time t minutes the temperature of the kettle is $T(t)$ degrees Celsius.

This temperature satisfies Newton's law of cooling

$$\frac{dT}{dt} = -k(T - \Theta)$$

where k is a positive constant.

After 5 minutes it is noted that $T = 69$, after 10 minutes $T = 51$ and after 15 minutes $T = 39$.

What was the initial temperature of the kettle and what is the temperature of the room? (7 marks)

Section E

- Attempt **all** questions in this section.
- It is suggested that you spend **approximately 36 minutes** on this section.
- You **must** show the method and workings you use to solve a question.
- Use a **separate answer booklet** for this section.
- This section assesses **Criterion 8.**

Question 25

If

$$\frac{z}{\sqrt{3}-i}=\frac{1}{\sqrt{3}+i}$$

determine the complex number z. Express your answer in Cartesian form. (3 marks)

Question 26

a) Solve $z^2+3z+9=0$ giving your answers in polar form. (4 marks)

b) If the roots of the equation are denoted z_1 and z_2 find all the integers N for which $z_1^N=z_2^N$.
(2 marks)

Question 27

a) Find the five roots of $z^5=4(1-i)$. (5 marks)

b) Sketch the locations of these roots in the complex plane. (3 marks)

Question 28

Consider the polynomial $P(z)$ given by

$$P(z)=z^4-8z^3+33z^2+\alpha z+\beta$$

in which α and β are real numbers.

a) Given that one root of $P(z)=0$ is $z=2+3i$, determine the values of α and β. (4 marks)

b) Hence, or otherwise, determine the other three roots of $P(z)=0$. (4 marks)

Section E continues

Question 29

Sketch on the one diagram the region of the Argand diagram defined by

$$1 \le |z + 2i| \le 2 \text{ and } \left|\arg(z + 4i) - \frac{3\pi}{8}\right| \le \frac{\pi}{8}.$$ (5 marks)

Question 30

a) Write down the sum of the geometric sequence

$$\sum_{n=0}^{6} z^n.$$ (1 mark)

b) Hence, or otherwise, determine the solutions of the polynomial equation

$$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0.$$ (3 marks)

c) Write the polynomial

$$z^6 + z^5 + z^4 + z^3 + z^2 + z + 1.$$

as the product of three quadratic polynomials with real coefficients. (2 marks)

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End of Exam

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