TASMANIAN ASSESSMENT, STANDARDS & CERTIFICATION

External Assessment 2023

Attach your candidate label here

MATHEMATICS SPECIALISED

MTS415118

Pages: 16
Questions: 30
Answer Booklets: 5
Information Sheet: 1

Preparation time for this exam: 15 minutes

Suggested working time: 3 hours

Instructions:

- There are **five (5)** sections to this exam paper.
- Answer **all** questions and **all** items within each question.
- You **must** show the methods you used to solve questions to receive full marks.
- Answer each section in a **separate answer booklet**.
- Approved calculators and all their functions may be used.
- The exam is **three (3)** hours in length. The suggested working time for each section is **approximately 36 minutes**.
- The Mathematics Specialised Information Sheet can be used throughout this exam.
- All answers must be written in **English**.
- You **must** make sure your answers address the listed criteria.

Guide to Exam Structure

	Questions available	Questions to answer	Suggested working time	Marks available
Section **A**	6	6	36 minutes	36 marks
Section **B**	6	6	36 minutes	36 marks
Section **C**	6	6	36 minutes	36 marks
Section **D**	6	6	36 minutes	36 marks
Section **E**	6	6	36 minutes	36 marks
Totals	**30**	**30**	**180 minutes (3 hours)**	**180 marks**

Criteria

You **must** make sure your answers address:

- Criterion 4 solve problems and use techniques involving finite and infinite sequences and series
- Criterion 5 solve problems and use techniques involving matrices and linear algebra
- Criterion 6 use differential calculus and apply integral calculus to areas and volumes
- Criterion 7 use techniques of integration and solve differential equations
- Criterion 8 solve problems and use techniques involving complex numbers.

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Section A

- Answer **all** questions in this section.
- The suggested working time for this section is **approximately 36 minutes**.
- This section assesses **Criterion 4**.

Question 1

The first, 5th and 13th terms of an arithmetic sequence with common difference $d \ (\neq 0)$ are the first three terms of a geometric sequence with common ratio $r \ (\neq 1)$. If the first term of both sequences is 12, find the values of d and r. (4 marks)

Question 2

Use mathematical induction to prove that

$$1^2 \times 2^1 + 2^2 \times 2^2 + 3^2 \times 2^3 + \ldots\ldots + n^2 \times 2^n = 2^{n+1}(n^2 - 2n + 3) - 6$$

for $n \geq 1$

(6 marks)

Question 3

Use formal methods to prove that the sequence

$$\frac{3n}{7n-4}$$

converges to $\frac{3}{7}$ as $n \to \infty$ (6 marks)

Question 4

Determine whether the following sequences $\{u_n\}$ converge or diverge.

Give a brief justification for your assertions **and** state the limit of any sequences that do converge.

a) $\cos\left\{\pi\left(n + \frac{1}{n}\right)\right\}$ (2 marks)

b) $e^{\left(\frac{(-1)^n n+1}{n^2+1}\right)}$ (2 marks)

c) $\left(1 + \frac{1}{n}\right)^n$ (2 marks)

Question 5

a) Verify that

$$(r+1)^4 - r^4 = 4r^3 + 6r^2 + 4r + 1$$ (1 mark)

b) Hence, or otherwise, use the method of differences to prove that

$$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$$ (6 marks)

(You may quote the formulae for $\sum_{r=1}^{n} r$ and $\sum_{r=1}^{n} r^2$)

Question 6

a) Derive the first four terms in the Maclaurin series of $\sqrt{1+x}$ (4 marks)

b) Use the Maclaurin series to obtain an approximate value for $\sqrt{1.01}$ (3 marks)

Section B

- Answer **all** questions in this section.
- The suggested working time for this section is **approximately 36 minutes**.
- This section assesses **Criterion 5**.

Question 7

Let M be the matrix given by

$$M = \begin{pmatrix} -3 & 2 \\ 4 & 5 \end{pmatrix}.$$

Determine the values of p and q if

$$M^2 = pM + qI$$

where I is the 2×2 identity matrix. (4 marks)

Question 8

The matrix A is given by

$$A = \begin{pmatrix} 2 & c \\ -1 & -1 \end{pmatrix}$$

where $c\ (\neq 2)$ is a constant.

a) Find A^{-1} (2 marks)

b) If $A + A^{-1} = I$, where I is the identity matrix, determine the value of c. (3 marks)

Question 9

The ellipse $2x^2 + 3y^2 = 1$ is rotated clockwise about the origin through an angle $\pi/3$ and then a shear of factor $\sqrt{3}$ in the y-direction is applied. Determine the equation of the image.

(7 marks)

Question 10

a) Given that the point P with co-ordinates $(1,-2,3)$ lies on the plane $\Pi: 3x+4y-2z=d$, determine the value of d. (2 marks)

b) Write down the equation of the line L that passes through P which is parallel to the direction $(2,\mu,-1)$. Express your answer in parametric form. (2 marks)

c) If the line L is embedded in the plane Π determine the value of μ. (2 marks)

Question 11

Consider the system of equations

$$\begin{aligned} 4x + 2y + \alpha z &= \beta \\ 3x + 2y + 16z &= 5 \\ x + y + 4z &= 1 \end{aligned}$$

a) Find the conditions on the constants α and β if this system has: (6 marks)

i. no solutions

ii. a unique solution, or

iii. an infinite number of solutions

b) Determine the general solution of the system when infinitely many solutions exist.

(2 marks)

Question 12

The transformation matrix T is defined to be

$$T = \begin{pmatrix} k & -2 \\ 1-k & k \end{pmatrix}$$

where k is a constant.

a) Determine the value of k if the image of every point on the line $y = 2x$ is also on this line. (3 marks)

b) If the unit square is transformed by T, for what value of k will the area of the image be least? (3 marks)

Section C

- Answer **all** questions in this section.
- The suggested working time for this section is **approximately 36 minutes**.
- This section assesses **Criterion 6**.

Question 13

a) Differentiate $x^2e^{\sin x}$ with respect to x. (2 marks)

b) Show that (3 marks)

$$\frac{d}{dx}\left(\ln\left\{e^x\left(\frac{x-2}{x+2}\right)^{3/4}\right\}\right)=\frac{x^2-1}{x^2-4}$$

Question 14

If $y=[\arcsin(x)]^2$ show that (5 marks)

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=2$$

Question 15

Consider the function

$$f(x)=\frac{x^2+3}{x-1}$$

a) Determine the intercepts and critical points of $f(x)$ (5 marks)

b) Sketch the graph of $y=f(x)$ (2 marks)

Question 16

Find the equation of the normal to the curve (6 marks)

$$x^2+8y^2=(2x^2+2y^2-x)^2$$

at the point (1,1).

Question 17

Determine the area in the first quadrant bounded by the parabola $y = 8 - x^2$ and the lines $y = 2x$ and $y = 7x$. (6 marks)

Question 18

The region bounded by the curve $x = 5 - 4y + y^2$ and the line $x = 5$ is rotated about the *y*-axis.

Find the volume of the solid of revolution formed. (7 marks)

Section D

- Answer **all** questions in this section.
- The suggested working time for this section is **approximately 36 minutes**.
- This section assesses **Criterion 7**.

Question 19

a) Determine $\frac{d}{dx}(x \arctan x)$ (2 marks)

b) Hence, or otherwise, evaluate (3 marks)

$$\int_0^1 \arctan x \, dx$$

Question 20

Use the substitution $u = \sqrt{x}$ to evaluate (5 marks)

$$\int_4^9 \frac{dx}{1+\sqrt{x}}$$

Question 21

Evaluate (5 marks)

$$\int_0^{\pi/2} x \cos x \sin x \, dx$$

Question 22

Find the general solution of the differential equation (7 marks)

$$(1+x^2)\frac{dy}{dx} + (1-y^2) = 0$$

Question 23

Determine the solution of the equation (7 marks)

$$xy^2\frac{dy}{dx} = x^3 + y^3 \qquad \text{given that } y = 0 \text{ when } x = e$$

Question 24

A suggested model for the growth of the number of cells C in a facial tumour that grows in a Tasmanian devil is provided by the Gompertz function $C(t)$ that satisfies the differential equation

$$\frac{dC}{dt} = C\ln(K/C)$$

where K is a constant.

a) Show that (1 mark)

$$\int \frac{dC}{C[\ln K - \ln C]} = \int dt$$

b) Use the substitution $v = \ln K - \ln C$ to solve for $C(t)$ given that $C = C_0$ when $t = 0$

(4 marks)

c) For what value of C is the number of cells growing at the greatest rate? (2 marks)

Section E

- Answer **all** questions in this section.
- The suggested working time for this section is **approximately 36 minutes**.
- This section assesses **Criterion 8**.

Question 25

Evaluate (4 marks)

$$\frac{1}{(2+i)^2} - \frac{1}{(2-i)^2}$$

Give your answer in polar form.

Question 26

Consider the two complex numbers

$$v = \frac{1}{5}(3+4i) \text{ and } w = \frac{1}{13}(5+12i) \text{ so that } |v| = |w| = 1$$

a) State the values of $|vw|$and $|v\overline{w}|$. (1 mark)

b) Calculate the complex numbers vw and $v\overline{w}$. (2 marks)

c) Hence, or otherwise, find two pairs of positive integers (m, n) such that $m^2 + n^2 = 4225(= 65^2)$. (2 marks)

Question 27

Determine the four roots of (5 marks)

$$z^4 = -1 - i\sqrt{3}$$

leaving your answers expressed in polar form with arguments in the range $(-\pi, \pi]$.

Question 28

a) Determine the real constants α and β if $z = 3i$ is one root of the equation (3 marks)

$P(z) = 0$ where

$$P(z) = z^4 + \alpha z^3 + \beta z^2 + 18z + 45$$

b) Hence solve $P(z) = 0$ giving all solutions in Cartesian form. (3 marks)

Question 29

a) Show that the set of points such that (4 marks)

$$|z + 1| = \sqrt{2}|z - i|$$

is a circle. State the position of its centre and its radius.

b) Show on an Argand diagram the set of points given by (4 marks)

$$\{z: |z + 1| \geq \sqrt{2}|z - i|\} \cap \{z: Re(z) \leq 0\}$$

Include on your sketch the co-ordinates of any significant points.

Question 30

a) Use de Moivre's theorem to show that (4 marks)

$$\sin 5\theta = 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta$$

b) Hence, or otherwise, prove that (4 marks)

$$\sin\left(\frac{\pi}{5}\right) = \frac{1}{4}\sqrt{10 - 2\sqrt{5}}$$

End of Exam

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