TASMANIAN ASSESSMENT, STANDARDS & CERTIFICATION

External Assessment 2024

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MATHEMATICS SPECIALISED

MTS415118

Pages: 16
Questions: 30
Answer Booklets: 5
Information Sheet: 1

Preparation time for this exam: 15 minutes

Suggested working time: 3 hours

Instructions:

- There are **five (5)** sections to this exam paper.
- Answer **all** questions and **all** items within each question.
- You **must** show the methods you used to solve questions to receive full marks.
- Answer each section in a **separate answer booklet**.
- Approved calculators and all their functions may be used.
- The exam is **three (3) hours** in length. The suggested working time for each section is **approximately 36 minutes**.
- The Mathematics Specialised Information Sheet can be used through this exam.
- All answers must be written in **English**.
- You **must** make sure your answers address the listed criteria.

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Guide to Exam Structure

	Questions available	Questions to answer	Suggested working time	Marks available
Section A	6	6	36 minutes	36 marks
Section B	6	6	36 minutes	36 marks
Section C	6	6	36 minutes	36 marks
Section D	6	6	36 minutes	36 marks
Section E	6	6	36 minutes	36 marks
Totals	**30**	**30**	**180 minutes (3 hours)**	**180 marks**

Criteria

You **must** make sure your answers address:

- Criterion 4 solve problems and use techniques involving finite and infinite sequences and series
- Criterion 5 solve problems and use techniques involving matrices and linear algebra
- Criterion 6 use differential calculus and apply integral calculus to areas and volumes
- Criterion 7 use techniques of integration and solve differential equations
- Criterion 8 solve problems and use techniques involving complex numbers.

Section A

- Answer **all** questions in this section in a **separate answer booklet**.
- This section is worth 36 marks and the suggested working time for this section is **approximately 36 minutes**.
- This section assesses **Criterion 4**.

Question 1

Find the real values of x for which $|2-9x| \geq 6$. (3 marks)

Question 2

Use mathematical induction to prove that:

$$1 \times 3^1 + 2 \times 3^2 + 3 \times 3^3 + \cdots + n \times 3^n = \tfrac{3}{4}[(2n-1)3^n + 1] \qquad \text{for } n \geq 1.$$ (6 marks)

Question 3

Use formal methods to prove that the sequence:

$$\frac{7n+5}{2n-9}$$

converges to $\frac{7}{2}$ as $n \to \infty$. (5 marks)

Question 4

a) Find integers a, b and c such that

$$\sum_{r=1}^{n} (5r-2)^2 = \frac{1}{6} n(an^2 + bn + c)$$

(3 marks)

b) Hence, or otherwise, determine the integer k if

$$\sum_{r=1}^{k} (5r-2)^2 = 94k^2 \ .$$

(3 marks)

Question 5

a) Use the method of differences to prove that

$$\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{n(3n+5)}{4(n+1)(n+2)}$$

(5 marks)

b) Hence, or otherwise, determine the smallest integer N such that

$$\sum_{r=N+1}^{\infty} \frac{1}{r(r+2)} < \frac{1}{10}.$$

(3 marks)

Question 6

a) Derive the first three non-zero terms in the Maclaurin series of $\ln(1+x)$. (3 marks)

b) Hence write down the first three terms in the Maclaurin series for $\ln\left(\frac{1}{1-2x}\right)$. (2 marks)

c) Use your series in b) to derive an approximate value for $\ln(1.5)$ correct to **three (3)** decimal places.

(3 marks)

Section B

- Answer **all** questions in this section in a **separate answer booklet**.
- This section is worth 36 marks and the suggested working time for this section is **approximately 36 minutes**.
- This section assesses **Criterion 5**.

Question 7

The matrix A is given by

$$\mathrm{A}=\begin{pmatrix} x & 2 \\ -5 & -1 \end{pmatrix}.$$

Given that the determinant of A is -2:

a) Find x. (2 marks)

b) Find A^{-1}. (2 marks)

Question 8

Let M be the matrix given by

$$\mathrm{M}=\begin{pmatrix} 6 & -2 \\ -4 & 1 \end{pmatrix}$$

and I is the 2×2 identity matrix.

a) Prove that

$$\mathrm{M}^2 = 7\mathrm{M} + 2\mathrm{I}.$$ (2 marks)

b) The transformation represented by M maps the point P onto the point Q.

If Q has the co-ordinates $(2k+8, 2k-5)$, find the co-ordinates of P. (3 marks)

Question 9

a) Determine the image of the circle $x^2+y^2=9$ under the transformation B given by

$$\mathrm{B}=\begin{pmatrix} 3 & 2 \\ 2 & -1 \end{pmatrix}.$$ (5 marks)

b) What is the area of the image? (1 mark)

Question 10

Two planes Π_1 and Π_2 are given by the equations

$$\Pi_1:\ x + 2y - 3z = 4 \qquad \text{and} \qquad \Pi_2:\ 2x + 4y + cz = d$$

for some constants c and d.

a) Show that the points $\mathrm{P}\ (3,2,1)$ and $\mathrm{Q}\ (c, \frac{d}{4}, -2)$ lie on the planes Π_1 and Π_2 respectively.

(2 marks)

b) For what value of c are the two planes parallel? (1 mark)

c) With this choice of c, prove that when $d = 8$ the point Q also lies on Π_1 (2 marks)

d) Hence, or otherwise, find the equation of a line in parametric form which is embedded in the plane Π_1. (2 marks)

Question 11

The transformation T is the result of a rotation through $90°$ anti-clockwise about the origin O, followed by a reflection in the line $y = -x$.

a) Determine the matrix that represents T. (4 marks)

b) Give a geometrical interpretation of T. (2 marks)

Question 12

a) Use Gauss-Jordan elimination to find the solution of the system of equations

$$\begin{aligned} 2x + y - z &= 2 \\ x + y + z &= 1 \\ 2x + 5y + 7z &= 1. \end{aligned}$$

(4 marks)

b) Find the value of q for which the planes

$$\begin{aligned} 2x + y - z &= 2 \\ x + y + z &= q \\ 4x + 5y + 7z &= 1 \end{aligned}$$

intersect.

Determine the equation of the line of intersection. (4 marks)

Section C

- Answer **all** questions in this section in a **separate answer booklet**.
- This section is worth 36 marks and the suggested working time for this section is **approximately 36 minutes**.
- This section assesses **Criterion 6**.

Question 13

Differentiate the following with respect to x:

a) $\arctan(2x^2)$ (2 marks)

b) $3^{2x} + x\cos\vartheta$, where $\vartheta = \sin x$ (3 marks)

Question 14

a) If $y = \left[x + \sqrt{x^2+1}\right]^p$, where $p \in \mathbb{R}$, show that

$$\sqrt{x^2+1}\,\frac{dy}{dx} = py$$

(2 marks)

b) Hence deduce that

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - p^2y = 0.$$ (3 marks)

Question 15

Sketch the curve

$$y = \frac{x+2}{(x-3)^2}.$$

Indicate clearly any stationary points, asymptotes and points of inflection. (8 marks)

Question 16

Find the equation of the normal to the curve

$$4y^2\left(3x^2 - y^2\right)^2 = \left(x^3 + y^3\right)^4$$

at the point (1,1). (4 marks)

Question 17

a) Sketch the area A defined to be the region bounded by the curves $y^2 = 1 + x$ and $y = x^3$ with $0 \leq y \leq 1$. (2 marks)

b) Determine this area. (4 marks)

Question 18

The area in the first quadrant contained between the curves $16y = x^2$ and $y = x^{2/3}$ is rotated completely about the $x - \text{axis}$ to form a volume V_1 .

The same area is rotated completely about the $y - \text{axis}$ to give a volume V_2 .

Prove that $V_1 = \frac{16}{35} V_2$. (8 marks)

Section D

- Answer **all** questions in this section in a **separate answer booklet**.
- This section is worth 36 marks and the suggested working time for this section is **approximately 36 minutes**.
- This section assesses **Criterion 7**.

Question 19

a) Determine

$$\int_0^{\pi/2} \sin 2x \cos 5x \, dx.$$

(2 marks)

b) Evaluate

$$\int_1^e x^2 \ln x \;\; dx.$$

(3 marks)

Question 20

Use the substitution $v = 1 + e^x$ to evaluate

$$\int_0^1 \frac{e^{3x}}{1+e^x} \;\; dx.$$

(4 marks)

Question 21

a) Verify that

$$\frac{d}{dx}\{2x^n(1-x)^{3/2}\} = [2n - (2n+3)x]x^{n-1}\sqrt{1-x}.$$

(1 mark)

b) Deduce that if

$$I_n = \int_0^1 x^n \sqrt{1-x} \, dx$$

then

$$I_n = \frac{2n}{3+2n} I_{n-1}.$$

(3 marks)

c) Evaluate

$$\int_0^1 x^3 \sqrt{1-x} \, dx.$$

(3 marks)

Question 22

Solve the differential equation

$$x^2 \frac{dy}{dx} - y = 1 \qquad \text{subject to} \qquad y(1) = 1. \qquad \text{(5 marks)}$$

Question 23

Determine the solution of the equation

$$2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2} \qquad \text{given that } y = 2 \text{ when } x = 1.$$

(7 marks)

Question 24

Billy and Poppy are conducting an experiment to monitor the percentage of burrows that are occupied by wombats on a nearby island.

It is thought that $W(t)$ % of the burrows are occupied t months after the start of the experiment where

$$\frac{dW}{dt} = rW\left(1 - \frac{W}{100}\right)$$

for some unknown constant $r\ (> 0)$.

Initially $W = W_0\ (> 0)$.

a) Prove that the solution of the equation subject to the boundary condition is

$$W(t) = \frac{100W_0}{W_0 + (100 - W_0)e^{-rt}}.$$

(You may either solve the equation directly or use differentiation to show that the given solution satisfies the differential equation.)

(4 marks)

b) Deduce that, irrespective of the value of W_0, as $t \to \infty$ all the burrows will become occupied. (1 mark)

c) Billy visited the island after 3 months and estimated that $W = 40$.

Poppy visited after 6 months and found that $W = 60$.

After how long will 90% of the burrows be occupied? (3 marks)

Section E

- Answer **all** questions in this section in a **separate answer booklet**.
- This section is worth 36 marks and the suggested working time for this section is **approximately 36 minutes**.
- This section assesses **Criterion 8**.

Question 25

If $z = 3 - 4i$ and $w = 2 + i$, determine the complex number v if

$$\frac{1}{v} = \frac{1}{z} + \frac{1}{\overline{w}}.$$

(3 marks)

Question 26

If $\arg(z - 5) = 2\pi/3$ find the minimum possible value for $|z|$. (4 marks)

Question 27

a) If $z = e^{i\theta}$ prove that

$$z^2 - z - \frac{1}{z} + \frac{1}{z^2} = 4\cos^2\theta - 2\cos\theta - 2$$ (2 marks)

b) Hence, or otherwise, find all solutions of the quartic equation

$$z^4 - z^3 + 2z^2 - z + 1 = 0$$

giving your answers in the form $a + ib$. (4 marks)

Question 28

a) Determine the real constants α and β if $z = -2 - 3i$ is one root of the equation $P(z) = 0$ where

$$P(z) = z^4 + z^3 + 8z^2 + \alpha z + \beta$$ (4 marks)

b) Hence solve $P(z) = 0$ giving all solutions in Cartesian form. (4 marks)

Question 29

a) Sketch the curve in the Argand plane on which

$$|\text{Im}\,(z-3i)|^2 + |\text{Re}\,(z-4i)|^2 = 9.$$ (2 marks)

b) Show on an Argand diagram the set of points given by

$$\{z : |z-4-2i| \leq 2\} \cap \left\{z: 0 \leq \arg(z-2-2i) \leq \tfrac{\pi}{4}\right\} \cap \{z : |z-4| \leq |z-6|\}.$$

(6 marks)

Question 30

a) Write down the three roots of the equation $z^3 = 1$ giving the non-real roots in the form $e^{i\theta}$ where $-\pi < \theta \leq \pi$. (2 marks)

b) If one of the non-real roots is denoted ω, show that the other non-real root is ω^2. (1 mark)

c) Verify that

$$1 + \omega + \omega^2 = 0.$$ (1 mark)

d) By using the result c) prove the following:

i.

$$\frac{\omega}{1+\omega} = -\frac{1}{\omega}.$$

(1 mark)

ii.

$$\frac{\omega^2}{1+\omega^2} = -\omega.$$

(1 mark)

iii.

$$\left(\frac{\omega}{1+\omega}\right)^k + \left(\frac{\omega^2}{1+\omega^2}\right)^k = 2(-1)^k \cos\frac{2k\pi}{3} \quad \text{where} \quad k \in \mathbb{N}.$$

(1 mark)

End of Exam

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