

External Assessment 2025

# MATHEMATICS SPECIALISED

MTS415118

Pages: 16

Questions: 30

Answer Booklets: 5

Information Sheet: 1

**Preparation time for this exam:** 15 minutes

**Suggested working time:** 3 hours

## Instructions:

- There are **five (5)** sections to this exam paper.
- Answer **all** questions and **all** items within each question.
- You **must** show the methods you used to solve questions to receive full marks.
- Answer each section in a **separate answer booklet**.
- Approved calculators and all their functions may be used.
- The exam is **three (3) hours** in length. The suggested working time for each section is **approximately 36 minutes**.
- The Mathematics Specialised Information Sheet can be used through this exam.
- All answers must be written in **English**.
- You **must** make sure your answers address the listed criteria.

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# Guide to Exam Structure

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		Questions available	Questions to answer	Suggested working time	Marks available
Section	<b>A</b>	6	6	36 minutes	36 marks
Section	<b>B</b>	6	6	36 minutes	36 marks
Section	<b>C</b>	6	6	36 minutes	36 marks
Section	<b>D</b>	6	6	36 minutes	36 marks
Section	<b>E</b>	6	6	36 minutes	36 marks
<b>Totals</b>		<b>30</b>	<b>30</b>	<b>180 minutes (3 hours)</b>	<b>180 marks</b>

## Criteria

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You **must** make sure your answers address:

- Criterion 4 solve problems and use techniques involving finite and infinite sequences and series
- Criterion 5 solve problems and use techniques involving matrices and linear algebra
- Criterion 6 use differential calculus and apply integral calculus to areas and volumes
- Criterion 7 use techniques of integration and solve differential equations
- Criterion 8 solve problems and use techniques involving complex numbers.

# Section A

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- Answer **all** questions in this section in a **separate answer booklet**.
  - This section is worth 36 marks. The suggested working time for this section is **approximately 36 minutes**.
  - This section assesses **Criterion 4**.
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## Question 1

Use mathematical induction to prove that

$$(1 \times 3) + (2 \times 4) + (3 \times 5) + \dots + n(n + 2) = \frac{1}{6}n(n + 1)(2n + 7) \quad \text{for } n \geq 1.$$

(6 marks)

## Question 2

The sum of the first  $n$  terms of a series is  $3n^2 + 2n$ .

a) Show that the  $n^{\text{th}}$  term is given by  $T_n = 6n - 1$ . (2 marks)

b) Show that, for some constants  $a, b$  and  $c$ ,

$$\sum_{r=1}^n T_r^2 = n(an^2 + bn + c)$$

State the values of  $a, b$  and  $c$ . (3 marks)

## Question 3

a) Use formal methods to prove that the sequence

$$\frac{n^3 + n^2}{2n^3 - 3}$$

converges to  $\frac{1}{2}$  as  $n \rightarrow \infty$ . (4 marks)

b) Does the sequence

$$\frac{n^3 + 2^n}{2n^3 - 3}$$

converge as  $n \rightarrow \infty$ ? Give brief reasoning to justify your assertion. (2 marks)

#### Question 4

a) Use the fact that  $(|x| - 1)^2 \geq 0$  to deduce that

$$\frac{2|x|}{1+x^2} \leq 1.$$

(1 mark)

b) For what values of  $x$  does

$$\frac{2|x|}{1+x^2} = 1?$$

(1 mark)

c) Give the sum of the infinite series

$$1 + \frac{2x}{1+x^2} + \frac{4x^2}{(1+x^2)^2} + \frac{8x^3}{(1+x^2)^3} + \dots$$

State for which values of  $x$  the summation is valid.

(4 marks)

#### Question 5

a) If

$$f(r) = \frac{1}{(r+1)(r+2)}$$

show that

$$f(r-1) - f(r) = \frac{2}{r(r+1)(r+2)}.$$

(2 marks)

b) Hence prove that

$$\sum_{r=1}^n \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}.$$

(3 marks)

c) Find the smallest integer  $N$  such that

$$\sum_{r=N+1}^{\infty} \frac{1}{r(r+1)(r+2)} < 0.01.$$

(2 marks)

### Question 6

a) Show that the first four terms in the Maclaurin expansion of  $(1 + x)^a$  are

$$(1 + x)^a = 1 + ax + \frac{1}{2!}a(a - 1)x^2 + \frac{1}{3!}a(a - 1)(a - 2)x^3 + \dots$$

(3 marks)

b) Determine the sum of the series

$$1 + \frac{5}{3} + \frac{5 \times 7}{3 \times 6} + \frac{5 \times 7 \times 9}{3 \times 6 \times 9} + \dots$$

(3 marks)

# Section B

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- Answer **all** questions in this section in a **separate answer booklet**.
  - This section is worth 36 marks. The suggested working time for this section is **approximately 36 minutes**.
  - This section assesses **Criterion 5**.
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## Question 7

Let  $C$  be the matrix given by

$$C = \begin{pmatrix} 3 & -7 \\ 2 & 1 \end{pmatrix}$$

and  $I$  be the  $2 \times 2$  identity matrix.

a) Prove that

$$C^2 = 4C - 17I.$$

(2 marks)

b) Find constants  $\lambda$  and  $\mu$  such that

$$C^3 = \lambda C + \mu I.$$

(3 marks)

## Question 8

The matrix  $Q$  is given by

$$Q = \begin{pmatrix} g & 9 \\ 4 & g \end{pmatrix}.$$

a) For what values of the constant  $g$  is this matrix invertible?

(2 marks)

A second matrix  $R$  is formed by adding  $k$  multiples of the first row of  $Q$  to the second row so that

$$R = \begin{pmatrix} g & 9 \\ 4 + kg & g + 9k \end{pmatrix}.$$

b) Prove that the determinants of  $Q$  and  $R$  are the same.

(2 marks)

### Question 9

A matrix  $S$  is given by

$$S = \sqrt{2} \begin{pmatrix} -1 & 4 \\ 5 & 3 \end{pmatrix}.$$

The effect of  $S$  is equivalent to some transformation  $X$  followed by another transformation

$$Y = \begin{pmatrix} 3 & 5 \\ 8 & -2 \end{pmatrix}.$$

Find the matrix for  $X$  and describe its effect geometrically. (5 marks)

### Question 10

The points  $P$  and  $Q$  have co-ordinates  $(0,2,5)$  and  $(4,-6,3)$  respectively.

A plane  $\Pi_1$  has the equation

$$\Pi_1: x + y - z = 2.$$

- Find the equation of the line  $L_1$  that passes through the points  $P$  and  $Q$ . (2 marks)
- Where does  $L_1$  meet  $\Pi_1$ ? (2 marks)
- Find the equation of a second plane  $\Pi_2$  that is parallel to  $\Pi_1$  and which passes through the midpoint of  $PQ$ . (2 marks)
- Another line  $L_2$  is embedded in  $\Pi_2$ . Find a possible equation for  $L_2$ . (2 marks)

### Question 11

A matrix  $M$  represents a reflection in the line  $y = \frac{x}{\sqrt{3}}$  followed by a dilation in the  $x$ -direction by factor  $d$  ( $> 0$ ).

- Find the matrix  $M$ . (4 marks)

The matrix  $M$  transforms the unit square to a parallelogram of area  $\frac{1}{2} d^2$ .

- What is the value of  $d$ ? (2 marks)

### Question 12

Consider the system of equations

$$x + 5y + z = -1$$

$$x - 2y - 2z = 0$$

$$2x + 3y + \alpha z = \beta$$

a) Solve the system when  $\alpha \neq -1$ . (5 marks)

b) Discuss the solution of the equations if  $\alpha = -1$ . (3 marks)

# Section C

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- Answer **all** questions in this section in a **separate answer booklet**.
  - This section is worth 36 marks. The suggested working time for this section is **approximately 36 minutes**.
  - This section assesses **Criterion 6**.
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## Question 13

If

$$y = \left(\frac{1+x}{1-x}\right)^r$$

prove that

$$\frac{dy}{dx} = \frac{2ry}{1-x^2}.$$

(3 marks)

## Question 14

If  $f(x) = \cos(\ln(1+x))$  show that

$$(1+x)^2 \frac{d^2f}{dx^2} + (1+x) \frac{df}{dx} + f = 0.$$

(5 marks)

## Question 15

Sketch the curve

$$y = x^2 e^{-x}.$$

Indicate clearly any stationary points and the  $x$  co-ordinates of any points of inflection.

(6 marks)

## Question 16

The curve C is defined by the equation

$$3x^2 + 2xy + 3y^2 - 4x + 4y = 20.$$

a) Show that the gradient of the curve is given by

$$\frac{dy}{dx} = \frac{2-3x-y}{2+x+3y}.$$

(3 marks)

b) Find any point(s) at which the tangent to C is parallel to the  $y$ -axis.

(4 marks)

### Question 17

a) Verify that

$$\frac{d}{dx} \left\{ \arcsin(x) + x(2x^2 - 1)\sqrt{1 - x^2} \right\} = 8x^2\sqrt{1 - x^2}.$$

(3 marks)

b) The curve defined by  $y^2 = x^4(1 - x^2)$  for  $x \geq 0$  is sketched in Figure 1.

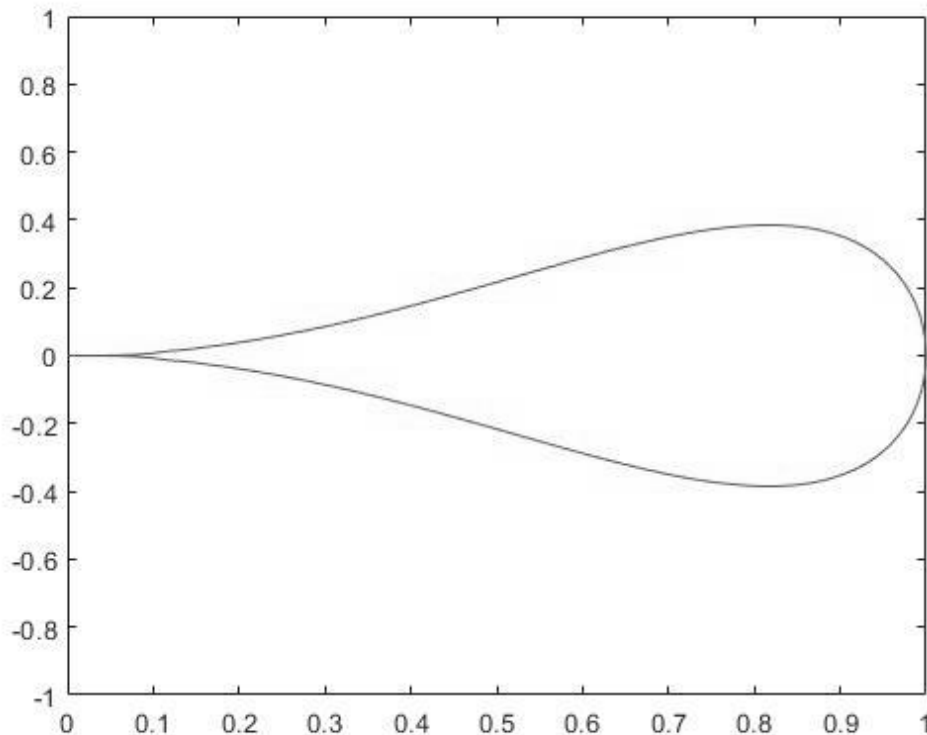


Figure 1

Determine the area enclosed by the curve in Figure 1.

(4 marks)

### Question 18

A region  $R$  of the  $(x, y)$  plane is bounded by the arc of the parabola  $8y = 15x^2 + \alpha$  ( $\alpha > 0$ ), the co-ordinate axes and the line  $x = 1$ .

The region  $R$  is rotated completely about the  $x$ -axis to form a volume  $V_1$ .

a) Find  $V_1$ .

(3 marks)

The same region  $R$  is then rotated completely about the  $y$ -axis to generate a volume  $V_2$ .

b) Find  $V_2$ .

(5 marks)

# Section D

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- Answer **all** questions in this section in a **separate answer booklet**.
  - This section is worth 36 marks. The suggested working time for this section is **approximately 36 minutes**.
  - This section assesses **Criterion 7**.
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## Question 19

a) Evaluate

$$\int_0^2 \frac{3 + 2s}{1 + 2s} ds.$$

(3 marks)

b) Determine

$$\int_0^{\pi/3} \sin^2 2\theta d\theta.$$

(3 marks)

## Question 20

Use the substitution  $v = x + 1$  to evaluate

$$\int_0^2 \frac{dx}{x^2 + 2x + 4}.$$

(4 marks)

## Question 21

a) If

$$I_n = \int_0^{\pi/2} x^n \sin x dx$$

show that

$$I_n + n(n-1)I_{n-2} = n \left(\frac{\pi}{2}\right)^{n-1}.$$

(4 marks)

b) Evaluate  $I_3$ .

(3 marks)

### Question 22

Solve the differential equation

$$x \cos y \frac{dy}{dx} + 1 = 0 \quad \text{subject to} \quad y(1) = \frac{\pi}{2}.$$

(5 marks)

### Question 23

Determine the general solution of the equation

$$\frac{dy}{dx} = \frac{2y(x+y)}{x(2x+y)}.$$

Express your answer in the form  $f(x, y) = C$ , a constant.

(6 marks)

### Question 24

A plane curve has the property that the gradient at any point P is the product of the gradients of the lines OP and AP, where O is the origin and A is the point  $(a, a)$ .

a) Explain why the defining differential equation for the curve is given by

$$\frac{dy}{dx} = \frac{y(a-y)}{x(a-x)}.$$

(2 marks)

b) Find the equation of the curve which has, as an asymptote, the line  $x = 2a$ .

(6 marks)

# Section E

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- Answer **all** questions in this section in a **separate answer booklet**.
  - This section is worth 36 marks. The suggested working time for this section is **approximately 36 minutes**.
  - This section assesses **Criterion 8**.
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## Question 25

Determine the complex number  $w$  that satisfies

$$\bar{w} = (2 + 3i)(w + 1).$$

(3 marks)

## Question 26

Two complex numbers  $z$  and  $w$  are represented by the points P and Q in an Argand diagram.

a) Prove that

$$|z + w|^2 + |z - w|^2 = 2\{|z|^2 + |w|^2\}.$$

(3 marks)

b) Give a geometrical interpretation of this result in terms of a parallelogram with adjacent sides OP and OQ. (2 marks)

## Question 27

a) Show that if  $z^8 + z^4 + 1 = 0$  then  $z^4 = e^{\pm \frac{2}{3}i\pi}$ .

(Hint: note that this is a quadratic equation in  $z^4$ ) (4 marks)

b) Hence deduce the eight roots of  $z^8 + z^4 + 1 = 0$ . (2 marks)

## Question 28

a) Determine the real constants  $\alpha$  and  $\beta$  if  $z = 2 + 3i$  is one root of the equation  $P(z) = 0$

where

$$P(z) = z^4 - 6z^3 + \alpha z^2 - 30z + \beta.$$

(4 marks)

b) Hence solve  $P(z) = 0$  giving all solutions in Cartesian form. (4 marks)

### Question 29

- a) Sketch the curve in the Argand plane on which

$$|z + 4| = |z - 3i|.$$

(3 marks)

- b) Show on an Argand diagram the set of points given by

$$\{z: |z - i| \geq 1\} \cap \{z: |z - 2i| \leq 2\} \cap \left\{z: \left| \operatorname{Arg}(z) - \frac{\pi}{2} \right| \geq \frac{\pi}{4}\right\}.$$

(5 marks)

### Question 30

- a) Use De Moivre's theorem to show that

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta.$$

(2 marks)

- b) Deduce that

$$\operatorname{cosec} 3\theta = \frac{\operatorname{cosec}^3 \theta}{3 \operatorname{cosec}^2 \theta - 4}.$$

(1 mark)

- c) By letting  $x = \operatorname{cosec} \theta$  and using b), find one solution of

$$x^3 - 6x^2 + 8 = 0.$$

(3 marks)

End of Exam



This exam paper and any materials associated with this exam  
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